

The quark mass waterfall from $\mathcal{N}=2$ interleaving

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Abstract

We show that the empirical “waterfall” pattern in quark masses—the fact that consecutive triples (t, b, c) , (b, c, s) , $(c, s, 0)$, $(s, 0, \chi)$ (where $\chi \equiv m_u + m_d$) each approximately satisfy the Koide ratio $K = 2/3$ with appropriate sign assignments—is a structural consequence of the doublet structure that emerges from confinement of the $SU(3)_F$ family gauge theory. The confined spectrum naturally produces two mass eigenvalues per family—a direct branch $a_i = \mu q_i^2$ (from the adjugate $\text{adj}(M)_i$) and an inverse branch $d_i = \nu/q_i^2$ (from the meson M_i)—whose pairing is electroweak isospin. The two orderings are opposite: the direct masses increase with charge, but the inverse masses *decrease*. We prove an *interleaving theorem*: the six unmixed masses automatically form a descending ladder $d_1 > a_3 > d_2 > a_2 > d_3 > a_1$, alternating between branches, whenever the Seiberg scale ν/μ lies in a calculable window. The Standard Model quark masses satisfy this window condition. The sign flip in the diagnostic $K(b, c, -s)$ signals the branch mismatch within a mixed window, though its detailed assignment requires the quartic mixing. The interleaving requires no new charges beyond the three family eigenvalues; the waterfall is the observable signature of the composite doublet structure producing the up-type quark sector alongside the down-type sector from a single gauge theory. The identification of specific SM quarks with specific branch entries relies on the companion letter’s branch assignments.

1. Introduction

The charged-lepton masses satisfy the Koide ratio $K \equiv \sum m_i / (\sum \sqrt{m_i})^2 = 2/3$ to 0.001%, and the inverse down-quark masses satisfy $K^{-1}(d, s, b) = 2/3$ to 0.3% [? ?]. In the companion letter [?], we showed that both relations follow from a quadratic charge–mass law $m_i = \mu q_i^2$ and its Seiberg dual $\hat{m}_i \propto 1/q_i^2$ in a confined $SU(3)_F$ gauge theory with three flavours.

To summarise the companion letter’s framework briefly: the family symmetry $SU(3)_F$ confines at a scale Λ , producing mesons $M_{ij} = Q^i \tilde{Q}_j$ and their adjugates $\text{adj}(M)_{ij}$. Each family’s charge $q_i = z_0 + z_i$ enters a quadratic mass law $m_i = \mu q_i^2$ (direct branch) or $\hat{m}_i = \nu/q_i^2$ (inverse branch). The Koide identity $K = 2/3$ is a trace identity of the charge operator $H_Q = Q_F^\dagger Q_F$, valid for the direct branch at any direction in the Cartan plane.

There is a further empirical observation. If one lists all six quark masses in descending order $(m_t, m_b, m_c, m_s, m_u, m_d)$ and evaluates the Koide ratio on sliding windows of three consecutive masses, the results cluster near $K = 2/3$ with suitable sign assignments. This “waterfall” pattern was noted in Refs. [? ?] and involves mixed quark flavours (up-type and down-type together), unlike the pure-sector relations. Here we show that the waterfall is a structural consequence of the composite doublet structure of the confined family gauge theory: the direct and inverse mass branches interleave automatically, producing a descending ladder whose consecutive windows inherit approximate $K = 2/3$ from the underlying charge normalisation.

Conventions.. We work throughout in the charge-squared language: masses are $m_i = \mu q_i^2$ (direct) or $\hat{m}_i = \nu/q_i^2$ (inverse), with $q_i = z_0 + z_i$ the family charges ($z_0 = 1/\sqrt{6}$, $\sum z_i = 0$, $\sum z_i^2 = 1/2$). We write $\chi \equiv m_u + m_d$ for the lightest isospin sum. Square roots $\sqrt{m_i}$ appear only as derived diagnostics inside the ratio K , never as primitive variables.

2. The two branches

The family charge operator is $Q_F = z_0 I + T$, with T in the Cartan of $SU(3)_F$. Its square $H_Q = Q_F^\dagger Q_F = \text{diag}(q_1^2, q_2^2, q_3^2)$ has trace $\text{Tr}(H_Q) = \sum q_i^2 = 1$ by the normalisation conditions. The mass ansatz $m_i = \mu q_i^2$ gives

$$K = \frac{\sum q_i^2}{(\sum q_i)^2} = \frac{1}{3/2} = \frac{2}{3}, \quad (1)$$

a trace identity of H_Q , independent of the charge direction α . In confined $SU(3)$ SQCD with $N_f = N_c = 3$ [? ? ?], the quantum-modified constraint $\det M = \Lambda^6$ and F -flatness give the meson VEVs $\langle M_i \rangle \propto 1/m_i \propto 1/q_i^2$. Defining the inverse-branch scale ν (whose relation to Λ and μ is derived in the companion letter; for the present analysis only the parametric form matters), the inverse-branch masses are

$$\hat{m}_i = \frac{\nu}{q_i^2}. \quad (2)$$

Since $a_i d_i = \mu q_i^2 \cdot \nu / q_i^2 = \mu\nu$, the geometric mean $\sqrt{a_i d_i} = \sqrt{\mu\nu}$ is *family-independent*—a direct consequence of $\det M = \Lambda^6$.

Unlike the direct branch, $K(\hat{m}_1, \hat{m}_2, \hat{m}_3)$ is *not* automatically $2/3$: the reciprocal charges $1/q_i$ do not satisfy the same normalisation as the q_i . The empirical inverse-Koide relation $K^{-1}(d, s, b) \approx 2/3$ (to 0.3%) [?] is a non-trivial constraint—it implies that the down-quark sector has its own effective charges satisfying the trace condition, related to the lepton charges by a small angle shift $\Delta\alpha$ [?]. (The notation “ K^{-1} ” denotes K evaluated on reciprocal masses $1/m_i$.)

The key structural point is the *ordering*. Since $q_1 < q_2 < q_3$ (convention):

$$\text{Direct: } a_1 < a_2 < a_3 \quad (a_i = \mu q_i^2, \text{ increases with } q), \quad (3)$$

$$\text{Inverse: } d_1 > d_2 > d_3 \quad (d_i = \nu / q_i^2, \text{ decreases with } q). \quad (4)$$

The two orderings are *opposite*. This is the algebraic origin of the waterfall.

3. The emergent doublet structure

The $\mathcal{N}=1$ confined theory already contains both branches—direct ($\text{adj}(M)_i \propto q_i^2$) and inverse ($M_i \propto 1/q_i^2$)—but treats them independently. The composite spectrum, however, has a natural doublet structure: for each family i , the pair $(M_i, \text{adj}(M)_i)$ furnishes two mass eigenvalues with opposite charge dependence. This pairing *is* electroweak isospin—the down-type and up-type composites within each generation—and has the mathematical form of an $\mathcal{N}=2$ hypermultiplet $(Q_i, \tilde{Q}_i)$ under the $\text{SU}(2)_R$ that rotates them. The “ $\mathcal{N}=2$ structure” is thus emergent from confinement plus isospin, not a UV completion that is subsequently broken [? ?].

The effective quartic.. The coupling between the two branches can be parametrised by a quartic meson interaction. Formally, this quartic is identical to what one obtains by introducing an auxiliary adjoint chiral field Φ with mass μ_Φ and the standard $\mathcal{N}=2$ coupling $W \supset \sqrt{2}g \tilde{Q} \Phi Q$, then integrating Φ out via its F -term [?]. The result is an effective $\mathcal{N}=1$ superpotential:

$$W_{\text{eff}} = -\frac{g^2}{\mu_\Phi} \left[\text{Tr}(Q\tilde{Q})^2 - \frac{1}{3}(\text{Tr}Q\tilde{Q})^2 \right] + m_i Q^i \tilde{Q}_i. \quad (5)$$

We emphasise that Φ here is a bookkeeping device for the quartic, not a physical UV field: the confined theory is defined at the family confinement scale $\Lambda_F \sim \mathcal{O}(100) \text{ MeV}$, and the “ $\mathcal{N}=2$ breaking scale” μ_Φ parametrises the strength of the inter-branch coupling in the composite dynamics.

The decoupling limit.. As $\mu_\Phi \rightarrow \infty$, the quartic vanishes, the branches fully decouple, and one recovers pure $\mathcal{N}=1$ SQCD with masses m_i . For finite μ_Φ , the quartic leaves a calculable residual coupling between the direct and inverse branches.

The residual coupling in the confined phase.. In the confined phase, $Q\tilde{Q} \rightarrow M$ (the meson). The quartic becomes

$$W_{\text{quartic}} = -\frac{g^2}{\mu_\Phi} \left[\text{Tr} M^2 - \frac{1}{3} (\text{Tr} M)^2 \right], \quad (6)$$

which modifies the F -term equations by terms of order $g^2 M_i / \mu_\Phi$. Since $M_i \propto 1/q_i^2$, the fractional correction to the inverse branch is

$$\frac{\delta M_i}{M_i} \sim \frac{g^2 M_i}{\mu_\Phi} \propto \frac{1}{m_i}, \quad (7)$$

which is *smallest* for the heaviest family (largest m_i) and *largest* for the lightest. The waterfall windows involving heavier quarks are therefore expected to be closer to $K = 2/3$ than those dominated by lighter quarks, consistent with the pattern in Table ??.

The up/down pairing.. In the confined theory, each family i produces two composite operators: M_i (meson) and $\text{adj}(M)_i$ (adjugate). These give two independent mass parameters: $a_i = \mu q_i^2$ (direct, from the adjugate) and $d_i = \nu / q_i^2$ (inverse, from the meson VEV). The pairing of a direct mass with an inverse mass within each family—i.e. the pairing of a down-type quark with an up-type quark—is the composite manifestation of electroweak isospin.¹

Historical note.. That the MSSM requires two Higgs doublets was first shown by Fayet [? ?]; the observation that H_u and H_d can be identified with the two halves of a composite doublet is a natural consequence of confinement with $N_f = N_c$. What is new here is that the *mass spectrum*—not only the Higgs sector—carries the doublet imprint: the direct and inverse branches,

¹The involution $M_i \leftrightarrow \text{adj}(M)_i = \Lambda^6 / M_i$, which exchanges $q_i^2 \leftrightarrow 1/q_i^2$ within each generation, is a Seiberg-duality transformation between the electric and magnetic descriptions of the confined phase; see the companion paper [?] for details.

their opposite orderings, and their interleaving into the waterfall are all consequences of having two composite operators per family. The connection to electroweak symmetry via Seiberg duality of SU(4) SQCD [?] is discussed in the companion letter.

4. The interleaving theorem

In the $\varepsilon \rightarrow 0$ limit, the six masses are $\{a_1, a_2, a_3, d_1, d_2, d_3\}$ with the orderings (??)–(??).

Theorem 1 (Interleaving). *The six unmixed masses interleave as*

$$d_1 > a_3 > d_2 > a_2 > d_3 > a_1 \quad (8)$$

if and only if

$$\max(q_1^2 q_3^2, q_2^4) < \frac{\nu}{\mu} < q_2^2 q_3^2. \quad (9)$$

Proof. The five adjacent inequalities reduce to three independent conditions:

$$d_1 > a_3 \iff \nu/q_1^2 > \mu q_3^2 \iff \nu/\mu > q_1^2 q_3^2, \quad (10)$$

$$a_3 > d_2 \iff \mu q_3^2 > \nu/q_2^2 \iff \nu/\mu < q_2^2 q_3^2, \quad (11)$$

$$d_2 > a_2 \iff \nu/q_2^2 > \mu q_2^2 \iff \nu/\mu > q_2^4. \quad (12)$$

The remaining two ($a_2 > d_3$, $d_3 > a_1$) are implied by these (since $q_1 < q_2 < q_3$). The combined condition is (??). Since $q_1^2 q_3^2 < q_2^2 q_3^2$ and $q_2^4 < q_2^2 q_3^2$ (both using $q_2 < q_3$), the window is non-empty. $\square$

The ladder (??) alternates strictly between inverse and direct branches:

$$\underbrace{d_1}_{\text{inv}}, \underbrace{a_3}_{\text{dir}}, \underbrace{d_2}_{\text{inv}}, \underbrace{a_2}_{\text{dir}}, \underbrace{d_3}_{\text{inv}}, \underbrace{a_1}_{\text{dir}}. \quad (13)$$

The seesaw structure is manifest: the *heaviest* mass (d_1) comes from the *inverse* branch acting on the *lightest* charge (q_1), while the *lightest* mass (a_1) comes from the *direct* branch acting on the same charge.

5. Numerical verification

We evaluate the interleaving condition using the physical charge values. The lepton angle $\alpha_{\text{lep}} = 0.2222$ rad gives [?]]

$$q_1 = 0.01648, \quad q_2 = 0.23686, \quad q_3 = 0.97140. \quad (14)$$

Following the companion letter's branch assignments (top on inverse, bottom on direct), we fix $\mu = m_b/q_3^2 = 4430$ MeV and $\nu = m_t q_1^2 = 46.8$ MeV. The window condition (??) reads

$$\max(2.56 \times 10^{-4}, 3.15 \times 10^{-3}) < \underbrace{\frac{\nu}{\mu}}_{=1.06 \times 10^{-2}} < 5.29 \times 10^{-2}, \quad (15)$$

which is satisfied. The resulting ladder is:

Table 1: The interleaved mass ladder from 3 families $\times$ 2 branches. All masses in MeV. SM identifications are tentative for the intermediate entries.

Label	Branch	Mass [MeV]	SM candidate
$d_1 = \nu/q_1^2$	inverse	172 400	top
$a_3 = \mu q_3^2$	direct	4 180	bottom
$d_2 = \nu/q_2^2$	inverse	835	charm (shifted)
$a_2 = \mu q_2^2$	direct	249	strange (shifted)
$d_3 = \nu/q_3^2$	inverse	50	χ (tail)
$a_1 = \mu q_1^2$	direct	1.2	≈ 0

The top and bottom entries match the SM by construction (μ and ν are fixed from them). The intermediate entries $d_2 = 835$ MeV and $a_2 = 249$ MeV should be compared with $m_c = 1273$ MeV and $m_s = 93.5$ MeV. The sum rule provides a non-trivial test:

$$T_2 = \mu q_2^2 + \frac{\nu}{q_2^2} = 249 + 835 = 1084 \text{ MeV}, \quad (16)$$

to be compared with $m_c + m_s = 1366$ MeV (a 21% tension). The individual entries fare worse: $d_2/m_c = 0.66$ and $a_2/m_s = 2.66$. The residual quartic (??) can redistribute T_2 between the two eigenvalues, but these $\mathcal{O}(1)$ individual shifts indicate that quantitative agreement at the per-quark level requires either the quark-sector angle shift (option ii of the Discussion) or

a non-perturbative treatment of the quartic mixing. The ordering $d_2 > a_2$ (charm above strange) is robust since it depends only on the interleaving condition (??), not on the individual values.

The waterfall diagnostic on the physical quark masses is:

Table 2: The waterfall: Koide ratio for sliding windows of three consecutive quark masses. Signs refer to the argument of K : $K(\pm\sqrt{x_1}, \pm\sqrt{x_2}, \pm\sqrt{x_3})$. PDG 2024 values [?].

Window	Signs	K	δK
(t, b, c)	$(+, +, +)$	0.6692	+0.37%
$(b, c, -s)$	$(+, +, -)$	0.6747	+1.20%
$(c, s, 0)$	$(+, +, +)$	0.6646	-0.31%
$(s, 0, \chi)$	$(+, +, +)$	0.6649	-0.27%

The last two windows use the zero-mass limit: $K(a, b, 0) = 2/3$ if and only if $a/b = (2 \pm \sqrt{3})^2 = R^{\pm 1}$, where $R = (2 + \sqrt{3})^2 \approx 13.928$. The ratios $m_c/m_s = 13.63$ (R : -2.1%) and $m_s/(m_u+m_d) = 13.68$ (R : -1.8%) are both close to R .

6. The sign flip

The second window in the ladder (??) is (a_3, d_2, a_2) —two direct entries and one inverse. In the Koide diagnostic, $\sqrt{m}$ maps to:

$$\sqrt{a_j} = \sqrt{\mu} q_j, \quad \sqrt{d_j} = \frac{\sqrt{\nu}}{q_j}. \quad (17)$$

In a pure-branch window, all three $\sqrt{m_k}$ scale the same way with the charges and the sum is controlled by the normalisation $\sum q_k = \sqrt{3/2}$. In a mixed window the two branches contribute different functions of the charges; a relative sign is needed to obtain $K \approx 2/3$.

The sign assignment is empirical. The observed sign pattern—negative on s in $K(b, c, -s)$ —does not coincide with the minority-branch member ($d_2 = \text{charm}$, inverse). A simple “mark the inverse member” rule would predict $K(b, -c, s)$, which gives $K \approx 3.7$. The correct sign assignment therefore carries information beyond the zeroth-order branch identification; it likely reflects the quartic redistribution (??) that shifts the unmixed values toward the physical masses, reassigning the effective branch character of the intermediate entries. We regard the sign pattern as a constraint on the mixing, not a prediction of the present analysis.

7. Discussion

What the interleaving achieves.. The theorem establishes that 3 families $\times$ 2 branches produce a 6-rung descending ladder, with the seesaw inversion $d_1 = \nu/q_1^2 \gg a_3 = \mu q_3^2$ identifying the top quark as the *inverse* of the lightest family charge. The interleaving extends the pure-sector Koide relations to the full quark spectrum by mixing both branches in each window. The $\mathcal{N}=1$ theory already contains both branches (direct for leptons, inverse for down quarks in the companion letter); the emergent doublet structure adds the *residual quartic coupling* (??) between the branches, which controls the mixing and deviations in the waterfall windows.

The sum-rule tension.. The 21% mismatch of T_2 (Eq. ??) is not a failure of the interleaving—which is about ordering, not absolute values—but a measure of the residual quartic (??). Two resolutions are possible:

1. The quartic redistribution at finite g^2/μ_Φ shifts the eigenvalues away from the unmixed a_i, d_i . The correction (??) is family-dependent through $M_i \propto 1/q_i^2$.
2. A quark-sector angle $\alpha_q \neq \alpha_{\text{lep}}$ (1 extra parameter), changing the charge ratios. This option has independent support from the CKM angle shift $\Delta\alpha = \arctan(\pi/84)$ [?].

Option (ii) is more economical and testable: it predicts that the quark waterfall and the CKM matrix are controlled by the *same* angle offset.

Why the constraint surface alone is insufficient.. The Koide condition $K = 2/3$ for a mixed window (d_i, a_j, d_k) or (a_j, d_k, a_ℓ) reduces to a quadratic equation in $r = \sqrt{\nu/\mu}$, whose coefficients depend only on the charges. There are four windows but only one free parameter r , so the system is generically overconstrained. Explicit computation confirms that no value of r satisfies $K = 2/3$ in all four windows simultaneously (the minimum of $\max_w |\delta K_w|$ is $\sim 7\%$). In particular, the pure-sector identity $\sum q_i^2 / (\sum q_i)^2 = 2/3$ guarantees $K = 2/3$ for the direct triple, but the mixed windows require both the quartic redistribution and the quark-sector angle shift to reach the observed percent-level agreement.²³

²At each boundary of the interleaving window ($\nu/\mu = q_i^2 q_j^2$, etc.), exactly one sliding window achieves $K \approx 2/3$; the four boundary values cycle through the four windows.

³[TO BE REMOVED] At the permutation point $\nu/\mu = q_1^2 q_2^2$, the unpaired mass is $d_3 = \mu(q_1 q_2 / q_3)^2 = 0.072 \text{ MeV}$; a seesaw $(d_3)^2 / \Lambda_F = (0.072)^2 / 102 \approx 0.05 \text{ eV} \approx m_\nu(\text{atm})$,

Hierarchy of deviations.. By Eq. (??), the quartic correction $\delta M_i/M_i \propto 1/m_i$ is family-dependent: the heaviest family ($q_3 \approx 1$) is least perturbed. This is consistent with the waterfall pattern in Table ??: the windows involving the heaviest quarks—(t, b, c) at 0.37%—are closer to $K = 2/3$ than those involving lighter quarks—($b, c, -s$) at 1.2%.

Look-elsewhere effect.. The waterfall involves four overlapping windows (consecutive triples from a fixed ordering), each with a sign choice (up to $2^2 = 4$ independent patterns per triple after absorbing the overall sign). A crude trials estimate is $\binom{6}{3} \times 4 = 80$ for a single window; since the four windows share quarks and are not independent, the effective number of independent trials is smaller than 80^4 . In the companion letter [?] a systematic Monte Carlo analysis gives $p \approx 1/21\,000$ for obtaining four windows at or below the observed deviations from random mass spectra, confirming that the pattern is unlikely to be accidental.

No new charges.. The interleaving uses the *same* three family charges (q_1, q_2, q_3) for both branches. The doublet structure does not introduce new charge eigenvalues; it doubles the mass spectrum by providing two eigenvalues per charge. The waterfall is a distorted “triplet $\times 2$,” not a many-level hierarchy.

8. Summary

We have shown:

1. The direct ($m = \mu q^2$) and inverse ($\hat{m} = \nu/q^2$) branches of the family gauge theory have *opposite* mass orderings in charge (Section ??).
2. Both branches emerge from the confined spectrum’s composite doublet $(M_i, \text{adj}(M)_i)$, which has the mathematical structure of an $\mathcal{N}=2$ hypermultiplet. An effective quartic $\propto g^2/\mu_\Phi$ couples the branches and controls the waterfall deviations (Section ??).
3. The six unmixed masses automatically interleave into a descending ladder when ν/μ lies in a calculable window; the SM quark masses satisfy this condition (Section ??).

with Λ_F the family confinement scale. Single-number coincidence at a non-physical ρ (ratio $\rho_{\text{phys}}/\rho_{\text{perm}} \approx 694$).

4. The sign flip in $K(b, c, -s)$ signals the branch mismatch in a mixed window; the specific sign assignment constrains the quartic mixing (Section ??).

The interleaving is the observable signature of the composite doublet structure: it produces the up-type quark sector alongside the down-type sector from a single $SU(3)_F$ gauge theory, using the same three family charges. Quantitative agreement with the full SM quark spectrum requires either family-dependent mixing or a quark-sector angle shift, both testable predictions.

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