

Lecture 11: The Koide Relation and the Dualised Standard Model

The Dualised Standard Model — Part III Lecture Notes

Lent Term 2027

Contents

1 Introduction and motivation

In Lectures 7–8, we constructed the dualised Standard Model: a magnetic dual of SU(3) QCD with $N_f = 6$ flavours, where 18 Higgs doublets emerge from the mes-Higgs field Φ_H and the fermion mass hierarchy arises from hierarchical VEVs among these doublets. The Yukawa coupling y is *universal* (a prediction of the duality), and the effective Yukawa matrix is

$$y_{ij}^{\text{eff}} = y \frac{\langle H_{ij} \rangle}{v}, \quad (1)$$

where H_{ij} are the Higgs doublet components and v is the electroweak VEV.

In this lecture we ask: *does the dualised SM predict or constrain the fermion mass eigenvalues?* We establish one solid result and explore two conjectures:

- (R) The Koide relation $Q = 3/2$ is algebraically equivalent to a **VEV condition** on the Higgs doublet VEV ratios—a reformulation that introduces no new parameters and follows from the universal Yukawa structure of Lecture 8.
- (C1) The *numerical value* $Q = 3/2$ coincides with the SU(5) Casimir ratio $C_2(\mathbf{10})/C_2(\mathbf{5}) = 2(N-2)/(N-1)$, which uniquely selects $N = 5$. Whether the Higgs potential enforces this value is *conjectural*.

The Koide angle $\delta = 2/9$ remains an empirically determined parameter; its origin is an open problem (see §??).

2 The Koide relation as a VEV condition

This section contains the lecture’s motivating reformulation. It requires only the universal Yukawa coupling of Lecture 8 and standard algebra. The observation that the Koide condition becomes a VEV constraint in universal-Yukawa models was first made in the yukawaon programme [?, ?, ?]; we present it here in the specific context of the dualised SM.

2.1 Setup: from VEVs to masses

From Lecture 8, the effective Yukawa coupling for the k -th generation in a given charge sector is

$$y_k^{\text{eff}} = y \frac{\langle H_k \rangle}{v} = y \frac{\mu_{0k}^2}{M_k^2}, \quad (2)$$

where μ_{0k}^2 is the mixing mass between the SM Higgs H_0 and the k -th dormant doublet, and M_k is the diagonal mass of the k -th doublet. The fermion mass is then

$$m_k = y_k^{\text{eff}} \frac{v}{\sqrt{2}} = y \frac{v}{\sqrt{2}} \frac{\mu_{0k}^2}{M_k^2}. \quad (3)$$

Define the *mixing ratio*:

$$x_k \equiv \frac{\mu_{0k}}{M_k}. \quad (4)$$

Then $m_k = m_0 x_k^2$ where $m_0 = y v / \sqrt{2}$ is the universal scale, and

$$\sqrt{m_k} = \sqrt{m_0} x_k. \quad (5)$$

2.2 The Koide condition on mixing ratios

The Koide parameter for three fermion masses m_1, m_2, m_3 is

$$Q = \frac{(\sqrt{m_1} + \sqrt{m_2} + \sqrt{m_3})^2}{m_1 + m_2 + m_3}. \quad (6)$$

Substituting (??):

$$Q = \frac{(x_1 + x_2 + x_3)^2}{x_1^2 + x_2^2 + x_3^2}. \quad (7)$$

The universal scale m_0 cancels. The Koide condition $Q = 3/2$ is a **pure constraint on the mixing ratios** $x_k = \mu_{0k}/M_k$. This is the key rewriting of this lecture: in any model with a universal Yukawa coupling and multiple Higgs doublets, the Koide relation reduces to a condition on the *Higgs sector* alone. The cancellation of m_0 is guaranteed by the scale invariance of Q (homogeneous of degree zero), not by any special property of the dualised SM.

Important

The Koide condition is independent of the universal Yukawa coupling y and the Higgs VEV v . It constrains only the *ratios* μ_{0k}/M_k of the Higgs sector parameters. This reformulation is exact. What follows in Sections 3–4 is conjectural.

2.3 Decomposition into common and traceless parts

Write $x_k = z_0 + z_k$ with

$$z_0 = \frac{x_1 + x_2 + x_3}{3}, \quad z_1 + z_2 + z_3 = 0. \quad (8)$$

Then:

$$Q = \frac{9z_0^2}{3z_0^2 + \sum_k z_k^2}. \quad (9)$$

The condition $Q = 3/2$ requires

$$z_0^2 = \frac{1}{3} \sum_{k=1}^3 z_k^2. \quad (10)$$

This is the **trace condition**: the squared common piece equals one-third of the sum of squared traceless pieces. It is identical to Koide's original 1983 constraint $\text{Tr}[U^2] = \text{Tr}[V^2]$ [?].

2.4 What the potential must do

The effective potential for the dormant Higgs doublets (Lecture 8) is

$$V_{\text{eff}} = \sum_k M_k^2 |H_k|^2 + \sum_k \mu_{0k}^2 (H_0^\dagger H_k + \text{h.c.}) + \text{quartic terms}. \quad (11)$$

Minimising with respect to $\langle H_k \rangle$ gives $x_k = \mu_{0k}/M_k$ as before. The Koide condition (??) is then a **constraint on the potential minimum**: the quartic terms must be structured so that the minimum satisfies $z_0^2 = \frac{1}{3} \sum z_k^2$.

We do *not* derive that the potential has this property. The question of whether the emergent quasi-SUSY of the magnetic theory constrains the quartic couplings sufficiently is the central open problem of this programme.

3 Conjecture C1: $Q = 3/2$ and the SU(5) Casimir ratio

3.1 The numerical coincidence

For $SU(N)$, the quadratic Casimir operators of the fundamental representation ($\square$) and the antisymmetric two-tensor ($\wedge^2 \square$) satisfy:

$$\frac{C_2(\wedge^2 \square)}{C_2(\square)} = \frac{2(N-2)}{N-1}. \quad (12)$$

This follows from $C_2(\wedge^p N) = p(N-p)(N+1)/(2N)$. Setting the ratio to $3/2$:

$$\frac{2(N-2)}{N-1} = \frac{3}{2} \implies N = 5. \quad (13)$$

This is the unique positive integer solution.

N	$C_2(\wedge^2 N)/C_2(N)$	
3	1	(degenerate)
4	4/3	
5	3/2	$= Q_{\text{Koide}}$
6	8/5	
∞	2	

3.2 Why $N = 5$?

In the dualised SM of Lectures 7–8, the flavour group is $SU(6)$ (from $N_f = 6$). The value $N = 5$ does not correspond to $SU(6)$ but to $SU(5) \subset SU(6)$. The five lightest quarks (u, c, d, s, b) form the fundamental of an $SU(5)$ flavour subgroup, with the top quark as the sixth flavour playing a special role.

This $SU(5)$ flavour structure is precisely the *sBootstrap* framework [?, ?], where the top is called the “elephant” and the five light quarks are “turtles.” The decomposition

$$SU(6)_{\text{flavour}} \supset SU(5)_{\text{turtle}} \times U(1)_{\text{top}} \quad (14)$$

is natural from the mass hierarchy alone ($m_t \gg m_b$), but the duality does not *derive* it: nothing in the CS framework selects $SU(5)$ over $SU(6)$ as the relevant Casimir group.

Important

The coincidence $C_2(\mathbf{10})/C_2(\mathbf{5}) = 3/2 = Q_{\text{Koide}}$ is a *mathematical fact* about $SU(5)$. Whether the Higgs potential minimum is determined by this Casimir ratio is a *conjecture without proof*. No derivation exists connecting the one-loop potential of the 18-doublet sector to the eigenvalue condition (??).

For this coincidence to be physically meaningful, there must exist a dynamical channel through which C_2 ratios in an $SU(5)$ flavour group influence the Higgs potential minimum. Candidate mechanisms include: one-loop corrections proportional to C_2 entering the quartic couplings, renormalisation-group fixed-point conditions, or representation-theoretic constraints on the potential structure. None of these has been investigated.

3.3 Dynkin index integers

If one takes the $SU(5)$ Casimir connection seriously, the Dynkin index ratios provide further numerical coincidences:

$$\frac{T(\mathbf{15})}{T(\mathbf{5})} = N + 2 = 7, \quad (15)$$

$$\frac{T(\mathbf{24})}{T(\mathbf{5})} = 2N = 10, \quad (16)$$

$$\frac{T(\mathbf{10})}{T(\mathbf{5})} = N - 2 = 3 = N_c. \quad (17)$$

The integers 7 and 10 appear in empirical “Z-ladder” mass relations, and $N - 2 = 3$ is the number of QCD colours. These coincidences are suggestive but do not constitute a derivation.

4 The Koide angle δ : an open problem

The Koide condition (??) constrains the sum $\sum z_k^2$ relative to z_0^2 , but does not fix the *individual* z_k values. The mass spectrum is parametrised by a single angle δ in the Z_3 representation:

$$x_k = z_0 \left(1 + \sqrt{2} \cos \left(\frac{2\pi k}{3} + \delta \right) \right), \quad k = 1, 2, 3. \quad (18)$$

The value $\delta = 2/9$ rad reproduces the charged lepton masses to 10^{-5} precision [?].

Neither the dualised SM nor the sBootstrap currently derives this value. Within the multi-Higgs framework, δ is determined by the relative magnitudes of the mixing masses μ_{0k} and the diagonal masses M_k . These originate from Planck-scale operators in the electric theory (Lecture 8), and their structure is not sufficiently constrained by the duality to fix δ .

Deriving δ from a dynamical principle—whether from modular flavour symmetries, conformal fixed-point dynamics, or some other mechanism—remains the central open problem of the Koide programme.

5 What $Q = 3/2$ determines and what it leaves free

The Koide condition $Q = 3/2$ is one constraint on three eigenvalues, leaving a two-parameter family: the overall scale m_0 and the angle δ . The scale m_0 sets the heaviest mass; δ sets the ratios.

With the empirical $\delta = 2/9$, the Koide formula predicts

$$\frac{m_e}{m_\mu} = 0.004836, \quad (\text{measured: } 0.004836) \quad (19)$$

$$\frac{m_\mu}{m_\tau} = 0.05946, \quad (\text{measured: } 0.05946) \quad (20)$$

Both agree to better than 0.002%. But this precision is the content of the original Koide formula [?]; it is not a prediction of the dualised SM, which so far provides $Q = 3/2$ as a VEV condition but does not derive δ .

6 The chain angle and colour

The Koide chain [?, ?] extends the lepton relation to quarks: $(0, s, c) \rightarrow (-s, c, b) \rightarrow (c, b, t)$, with each link satisfying $Q \approx 3/2$ and the chain angle $\delta_{scb} \approx 3\delta_\ell$.

One interpretation of the factor 3: the quark sector carries $SU(3)_c$ colour, which multiplies the vacuum angle:

$$\delta_{\text{chain}} = N_c \delta_\ell = 3 \times \frac{2}{9} = \frac{2}{3} \text{ rad} = 38.2^\circ. \quad (21)$$

The measured value $\delta_{scb} \approx 37.2^\circ$ agrees to 2.7%. No calculation within the dualised SM or the sBootstrap currently produces the factor N_c from dynamics. Whether the 2.7% agreement is evidence for the colour interpretation or a numerical coincidence cannot be decided without a derivation of the vacuum structure in the coloured sector.

7 The inverse Koide from the seesaw

7.1 Q is preserved under inversion

The Seiberg seesaw (Lecture 3) gives the meson VEV $\langle M_{kk} \rangle = \Lambda^2/m_k$. Since Q is homogeneous of degree zero:

$$Q\left(\frac{\Lambda^2}{m_1}, \frac{\Lambda^2}{m_2}, \frac{\Lambda^2}{m_3}\right) = Q\left(\frac{1}{m_1}, \frac{1}{m_2}, \frac{1}{m_3}\right). \quad (22)$$

This is a theorem, not a conjecture.

7.2 Empirical status

With PDG 2024 pole masses for leptons and $\overline{\text{MS}}$ masses at $\mu = 2 \text{ GeV}$ for quarks [?]:

$$Q(m_e, m_\mu, m_\tau) = 1.50001, \quad Q(1/m_d, 1/m_s, 1/m_b) = 1.505 \pm 0.010. \quad (23)$$

The uncertainty on $Q(1/m)$ is dominated by $m_d = 4.70 \pm 0.20 \text{ MeV}$ (4.3%). The deviation from $3/2$ is 0.005 ± 0.010 , i.e. within 1σ : the agreement cannot currently be distinguished from numerical coincidence at this level of quark-mass precision.

Note that Q -preservation under inversion is a mathematical identity holding for any mass triplet; the empirical near-agreement $Q(1/d, 1/s, 1/b) \approx Q(e, \mu, \tau) \approx 3/2$ is a separate observation requiring an independent dynamical explanation.

The *individual mass ratios* do not match through a simple seesaw $m_\ell = c/m_q$ (see §??).

7.3 The hierarchy problem of the seesaw

The seesaw $m_\ell \propto 1/m_q$ gives an *inverted* hierarchy: the lightest quark maps to the heaviest lepton. With the standard generation pairing ($d \leftrightarrow e$, $s \leftrightarrow \mu$, $b \leftrightarrow \tau$): $m_e/m_\mu = m_s/m_d = 19.9$, while the actual ratio is 0.0048—a factor of 4000 discrepancy.

With the reversed pairing ($d \leftrightarrow \tau$, $s \leftrightarrow \mu$, $b \leftrightarrow e$): m_τ is recovered by construction, m_μ is 15% off, and m_e is a factor 4 wrong.

The seesaw preserves Q (that is exact) but does *not* reproduce the individual lepton masses. The quark–lepton mass mapping is more complex than a simple inversion.

8 Connection to the sBootstrap

The sBootstrap framework [?, ?] identifies sfermions with known mesons and diquarks via $SU(5)$ flavour symmetry. The closure conditions [?]

$$2N = k_u k_d, \quad 2N = \frac{k_d(k_d + 1)}{2}, \quad 4N = k_u^2 + k_d^2 - 1 \quad (24)$$

uniquely give $N = 3$ generations with $k_u = 2$, $k_d = 3$.

In the dualised SM, the ISS rank condition [?] (see also Lecture 6) provides a *dynamical* derivation of the same split.

8.1 From $N_f = 6$ to $N_f = 5$: the top quark as a trigger

Start with the full theory: $SU(3)_c$ with $N_f = 6$ flavours. The Seiberg dual is $SU(N_f - N_c) = SU(3)$ with 6 flavours and a 6×6 meson matrix $M_{ij} = \tilde{Q}_i Q_j$ containing 36 mesonic composites. The ISS rank condition at $N_f = 6$ gives $\text{rank}(\tilde{\varphi}\varphi) \leq N_f - N_c = 3$, producing a *symmetric* split: $k_u = k_d = 3$.

Now give one flavour (the top) a large mass m_t . In the magnetic theory, this corresponds to a VEV for the meson component M_{tt} , which Higgses the dual gauge group:

$$SU(3)_{\text{dual}} \xrightarrow{\langle M_{tt} \rangle \neq 0} SU(2)_{\text{dual}}. \quad (25)$$

Below the scale m_t , the effective theory has $N_f = 5$ active flavours, the dual gauge group is $SU(2)$, and the meson matrix reduces to $5 \times 5 = 25$ components. The ISS rank condition on the $SU(2)$ theory now gives $\text{rank}(\tilde{\varphi}\varphi) \leq N_f - N_c = 2$:

$$k_u = N_f - N_c = 2 \quad (\text{massless}), \quad k_d = N_c = 3 \quad (\text{massive}). \quad (26)$$

The **asymmetric 2 + 3 split is driven by the top mass**: it is the large m_t that Higgses $SU(3) \rightarrow SU(2)$ and breaks the $3 + 3$ symmetry of the $N_f = 6$ theory.

8.2 The meson decomposition

The 5×5 meson matrix transforms as $\mathbf{5} \otimes \bar{\mathbf{5}} = \mathbf{24} \oplus \mathbf{1}$ under $SU(5)$ flavour. The ISS 2 + 3 split decomposes $SU(5) \rightarrow SU(2)_u \times SU(3)_d \times U(1)$, and the 24 adjoint mesons organise by electric charge:

Block	$SU(2) \times SU(3)$ rep	Charge	Count
$\bar{u} u$ (traceless)	$(\mathbf{3}, \mathbf{1})$	0	3
$\bar{d} d$ (traceless)	$(\mathbf{1}, \mathbf{8})$	0	8
U(1) generator	$(\mathbf{1}, \mathbf{1})$	0	1
$\bar{d} u$	$(\mathbf{2}, \bar{\mathbf{3}})$	+1	6
$\bar{u} d$	$(\bar{\mathbf{2}}, \mathbf{3})$	-1	6

giving $\mathbf{24} \rightarrow 12_0 + 6_{+1} + 6_{-1}$, plus the flavour-singlet η' -like trace for a total of 25.

The sBootstrap [?, ?] identifies these 25 composites with the scalar partners of known fermions: the charged mesons ($\bar{d}_i u_j$ and $\bar{u}_i d_j$) map to charged sfermions, while the neutral composites map to sneutrinos and neutral Higgs-like states. The closure conditions then count:

$$2N = k_u k_d = 6, \quad 2N = \frac{k_d(k_d+1)}{2} = 6, \quad 4N = k_u^2 + k_d^2 - 1 = 12 \quad (27)$$

all giving $N = 3$ generations *uniquely*.

8.3 Koide from the meson mass formula

The sBootstrap mass formula $m_k = (z_0 + z_k)^2 K_\Omega$ is algebraically equivalent to the Koide condition (??), as shown in [?]. The dualised SM provides the *physical context* for this formula: $z_0 + z_k$ is the mixing ratio x_k of the Higgs sector, and K_Ω is the universal scale m_0 .

9 Summary and honest assessment

Result	Origin	Status
Koide = VEV condition	Universal Yukawa + algebra	Reformulated
$k_u = 2, k_d = 3$	ISS rank + top Higgsing	Derived
$Q(1/d, 1/s, 1/b) \approx 3/2$	Seesaw preserves Q	Theorem
$N_{\text{gen}} = 3$	Closure = duality constraint	Two derivations
$Q = C_2(\mathbf{10})/C_2(\mathbf{5})$	SU(5) Casimir	Observation
$\delta_{scb} = N_c \delta_\ell$	Empirical (factor 3)	2.7% match
$\delta = 2/9$	—	Open problem
VEV condition mechanism	Quartic structure	<i>Not derived</i>
Quark–lepton mapping	Seesaw details	<i>Fails quantitatively</i>
Z-ladder formula	—	<i>Not derived</i>

Scale convention. The VEV condition is imposed at the scale where the dormant Higgs doublets are integrated out (near the duality scale $\Lambda \sim \text{TeV}$ of Lecture 7). Below that scale, RG running introduces corrections that spoil $Q = 3/2$ for quark masses at low energies. The charged-lepton Koide relation holds for *pole masses*; the quark chain uses $\overline{\text{MS}}$ running masses at mixed scales. This difference is a known subtlety [?] and is unresolved in the present framework.

What is established. The universal Yukawa coupling of the dualised SM translates the Koide relation into a VEV condition on the multi-Higgs sector. The ISS rank condition derives the turtle/elephant decomposition. The seesaw preserves Q as a mathematical identity.

What is observed but not derived. The numerical coincidence $Q = C_2(\mathbf{10})/C_2(\mathbf{5}) = 3/2$ is an exact group-theoretic fact that matches the Koide value. It has no known dynamical origin within the dualised SM. The angle $\delta = 2/9$ has no known derivation at all.

What remains open. The quartic coupling structure of the 18-Higgs potential must be shown to have a minimum at the Koide point. The quark–lepton mass mapping through the seesaw preserves Q but not the individual mass ratios.

10 Exercises

Exercise 10.1. Verify that $C_2(\wedge^2 \mathbf{N})/C_2(\mathbf{N}) = 2(N-2)/(N-1)$ using the general formula $C_2(\wedge^p \mathbf{N}) = p(N-p)(N+1)/(2N)$. For which values of N is this ratio an integer?

Exercise 10.2. Compute Q for the three charged lepton pole masses ($m_e = 0.511 \text{ MeV}$, $m_\mu = 105.658 \text{ MeV}$, $m_\tau = 1776.93 \text{ MeV}$). Compare with $Q(1/m_d, 1/m_s, 1/m_b)$ using the PDG values $m_d = 4.70$, $m_s = 93.5$, $m_b = 4183 \text{ MeV}$.

Exercise 10.3. In the Z_3 parametrisation (??), show that $Q = 3/2$ is satisfied for *any* value of δ . Then verify that $\delta = 2/9$ reproduces the lepton mass ratios m_e/m_μ and m_μ/m_τ to better than 0.01%.

Exercise 10.4. The sBootstrap closure conditions give $k_u = 2, k_d = 3$ uniquely. The ISS rank condition gives $k_u = N_f - N_c, k_d = N_c$ for SQCD with N_f flavours and N_c colours. For which values of (N_c, N_f) do the two derivations agree?

Exercise 10.5 (Challenge). Write the most general quartic potential for three complex scalar doublets H_1, H_2, H_3 with an S_3 permutation symmetry softly broken by the mixing masses μ_{0k} . Does the minimum satisfy (??) for any choice of quartic couplings? What additional conditions (if any) are needed?

References

- [1] G. Cacciapaglia and F. Sannino, “Charting Standard Model Duality and its Signatures,” arXiv:2407.17281 [hep-ph] (2024).
- [2] A. Rivero, “Unbroken supersymmetry without new particles,” arXiv:0910.4793 [hep-ph] (2009).
- [3] A. Rivero, “An interpretation of scalars in $SO(32)$,” Eur. Phys. J. C **84**, 1058 (2024), arXiv:2407.05397 [hep-ph].
- [4] Y. Koide, “New view of quark and lepton mass hierarchy,” Phys. Rev. D **28**, 252 (1983).
- [5] C. Brannen, “The lepton masses,” <https://brannenworks.com/MASSES2.pdf> (2006).
- [6] S. Antusch, K. Hinze and S. Saad, “Updated Running Quark and Lepton Parameters at Various Scales,” arXiv:2510.01312 [hep-ph] (2025).
- [7] N. Seiberg, “Electric–magnetic duality in supersymmetric non-abelian gauge theories,” Nucl. Phys. B **435**, 129 (1995).
- [8] Y. Sumino, “Family gauge symmetry as an origin of Koide’s mass formula,” JHEP **0905**, 075 (2009).
- [9] N. Haba and Y. Koide, “New origin of a bilinear mass matrix form,” Phys. Lett. B **659**, 260 (2008), arXiv:0708.3913 [hep-ph].
- [10] E. Ma, “Lepton family symmetry and possible application to the Koide mass formula,” Phys. Lett. B **649**, 287 (2007), arXiv:hep-ph/0612022.
- [11] K. Intriligator, N. Seiberg and D. Shih, “Dynamical SUSY breaking in meta-stable vacua,” JHEP **0604**, 021 (2006), arXiv:hep-th/0602239.
- [12] A. Rivero, “A new Koide tuple: strange-charm-bottom,” arXiv:1111.7232 [hep-ph] (2011).
- [13] P. Zenczykowski, “Remark on Koide’s Z_3 -symmetric parametrization of quark masses,” Phys. Rev. D **86**, 117303 (2012), arXiv:1210.4125 [hep-ph].