The Koide relation from two Casimir ratios: $Q = C_2(\wedge^2 5)/C_2(5)$, $\delta = C_2(3)/C_2(S^3 3)$

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Abstract

Both the Koide ratio $Q = 3/2$ and the Koide angle $\delta = 2/9$ are identified as quadratic Casimir ratios of simple Lie groups. (1) $Q = C_2(\wedge^2\mathbf{5})/C_2(\mathbf{5}) = 3/2$ in $\mathrm{SU}(5)_{\text{flavour}}$ (the sBootstrap turtle group), uniquely selecting $N_f = 5$. (2) $\delta = C_2(\mathbf{3})/C_2(S^3\mathbf{3}) = (4/3)/6 = 2/9$ in $\mathrm{SU}(3)_{\text{generation}}$ (three generations embedded in $G_2 = \mathrm{Aut}(\mathbb{O})$). The chain angle factor $\delta_{scb} \approx 3\delta_\ell$ is the colour multiplicity $N_c = 3$. Together, these determine the charged lepton mass ratios from group theory alone: Q constrains the trace ratio (how much masses spread) and δ constrains the angular position (where they sit on the Koide orbit). The sBootstrap provides the physical context: $\mathrm{SU}(5)_{\text{flavour}}$ acts on the five turtle quarks, $\mathrm{SU}(3)_{\text{generation}}$ acts on the three generations, and $N_c = 3$ is QCD colour. No new parameters are introduced.

1 The two Casimir identities

1.1 $Q = 3/2$ from SU(5)

The Koide ratio $Q \equiv (\sum\sqrt{m_i})^2/\sum m_i$ equals the quadratic Casimir ratio of the antisymmetric two-tensor to the fundamental of $\mathrm{SU}(N)$:

$$Q = \frac{C_2(\wedge^2\mathbf{N})}{C_2(\mathbf{N})} = \frac{2(N-2)}{N-1}. \tag{1}$$

Setting $Q = 3/2$: $N = 5$. In the sBootstrap, SU(5) is the flavour group of the five turtle quarks (u, c, d, s, b).

Further SU(5) invariants give Z-ladder integers: $T(\mathbf{15})/T(\mathbf{5}) = N + 2 = 7$, $T(\mathbf{24})/T(\mathbf{5}) = 2N = 10$.

1.2 $\delta = 2/9$ from SU(3) $\subset G_2$

The Koide angle $\delta = 2/9$ equals the Casimir ratio of the fundamental to the symmetric three-tensor of SU(3):

$$\delta = \frac{C_2(\mathbf{3}_{SU(3)})}{C_2(S^3\mathbf{3}_{SU(3)})} = \frac{4/3}{6} = \frac{2}{9}. \tag{2}$$

Here SU(3) is the *generation* symmetry (three generations of fermions), naturally embedded in $G_2 = \mathrm{Aut}(\mathbb{O})$ (the automorphism group of the octonions, which provides the exceptional Jordan algebra $J_3(\mathbb{O})$ as a three-generation mass matrix framework).

The symmetric three-tensor $S^3(\mathbf{3})$ has dimension $\binom{5}{3} = 10$ and Casimir $C_2 = p(N+p)(N-1)/(2N)|_{N=3,p=3} = 6$.

2 The two groups

Group	Acts on	Casimir ratio	Result
$\mathrm{SU}(5)_{\text{flavour}}$	5 turtle quarks	$C_2(\wedge^2)/C_2(\text{fund}) = 3/2$	Q_{Koide}
$\mathrm{SU}(3)_{\text{generation}}$	3 generations	$C_2(\text{fund})/C_2(S^3) = 2/9$	δ_{Koide}

The two groups act on *different spaces*: SU(5) mixes flavours (u, c, d, s, b), SU(3) mixes generations $(1, 2, 3)$. Their Casimir ratios determine orthogonal aspects of the mass spectrum: Q (the trace condition) and δ (the angular position).

3 The chain angle as colour multiplicity

The chain angle $\delta_{scb} \approx 37.2° \approx 0.649$ is close to $3\delta_\ell = 3 \times 2/9 = 2/3 \approx 38.2°$ (deviation 1.0°). The factor 3 is N_c, the number of QCD colours:

$$\delta_{scb} \approx N_c \cdot \delta_\ell. \tag{3}$$

This connects the quark Koide chain to the lepton Koide through the colour multiplicity, completing the group-theoretic picture:

$$Q \leftarrow \mathrm{SU}(5)_{\text{flav}}, \qquad \delta \leftarrow \mathrm{SU}(3)_{\text{gen}}, \qquad \delta_{scb}/\delta_\ell \leftarrow \mathrm{SU}(3)_{\text{colour}}. \tag{4}$$

4 What is and is not derived

Quantity	Formula	Status
$Q = 3/2$	$C_2(\wedge^2\mathbf{5})/C_2(\mathbf{5})$	Group theory (exact)
$\delta = 2/9$	$C_2(\mathbf{3})/C_2(S^3\mathbf{3})$	Group theory (exact)
$N = 5$	$2(N-2)/(N-1) = 3/2$	Uniquely selected
$\delta_{scb}/\delta_\ell = 3$	N_c	Colour multiplicity (1° off)
Why mass matrix $\sim$ Casimirs	—	Not derived
Z-ladder formula	—	Not derived
Overall mass scale M_0	—	Free parameter

The Casimir identities provide the *values* of Q and δ from group theory, and *select* $N = 5$ and $N_{\text{gen}} = 3$ as the only viable integers. The *mechanism* that forces the mass matrix to satisfy these Casimir ratios remains the key open question. Candidates include: a conformal fixed point where anomalous dimensions are Casimir-scaled; a vacuum alignment condition in the Higgs potential; or a string-theoretic construction where the octonion structure of G_2 determines the mass matrix.

References

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[2] A. Rivero, Eur. Phys. J. C **84**, 1058 (2024).