Exact Koide Chain from $\delta = 15°$

Notebook – phys6gpd

1 Tuple 1: $(0,\, q_2^2,\, q_3^2)$ with $\delta = 15°$

The Koide parametrization writes the three charges (square roots of masses) as

$$q_k = \sqrt{M_0}\Big(1 + \sqrt{2}\,\cos\big(\tfrac{2\pi k}{3} + \delta\big)\Big), \qquad k = 1, 2, 3.$$

Setting $\delta = \pi/12 = 15°$, the $k\!=\!1$ entry vanishes:

$$\cos\big(\tfrac{2\pi}{3} + \tfrac{\pi}{12}\big) = \cos\tfrac{3\pi}{4} = -\tfrac{1}{\sqrt{2}},$$

so $q_1 = \sqrt{M_0}(1-1) = 0$. The surviving charges are

$$q_2 = \sqrt{M_0}\Big(1 + \sqrt{2}\cos\tfrac{17\pi}{12}\Big) = \sqrt{M_0}\,\frac{3-\sqrt{3}}{2}, \tag{1}$$

$$q_3 = \sqrt{M_0}\Big(1 + \sqrt{2}\cos\tfrac{\pi}{12}\Big) = \sqrt{M_0}\,\frac{3+\sqrt{3}}{2}. \tag{2}$$

The unnormalised masses are q_k^2; their sum is

$$q_2^2 + q_3^2 = M_0\left(\frac{(3-\sqrt{3})^2 + (3+\sqrt{3})^2}{4}\right) = M_0\,\frac{24}{4} = 6\,M_0.$$

Setting $\sqrt{M_0} = 1$ henceforth, the Koide scale parameter is $Q_0^{(1)} = (0 + q_2^2 + q_3^2)/3 = 2$.

Normalising so that $0 + x^2 + y^2 = 1$ (with $x < y$):

$$\boxed{x^2 = \frac{2-\sqrt{3}}{4} \approx 0.0670, \qquad y^2 = \frac{2+\sqrt{3}}{4} \approx 0.9330.}$$

These correspond to m_s and m_c in the (u, s, c) triplet of the Koide chain, with $m_u = 0$.

2 Tuple 2: $(-q_2,\, q_3,\, z)$ — predicting the b charge

The next link in the chain takes charges $(-q_2,\, q_3,\, z)$ with $z > 0$, corresponding to the (s, c, b) triplet with the sign flip $-\sqrt{m_s}$. The Koide condition reads

$$\frac{(-q_2 + q_3 + z)^2}{q_2^2 + q_3^2 + z^2} = \frac{3}{2}.$$

2.1 Key intermediate: $q_3 - q_2 = \sqrt{3}$

$$q_3 - q_2 = \frac{(3+\sqrt{3}) - (3-\sqrt{3})}{2} = \sqrt{3}, \qquad q_2^2 + q_3^2 = 6.$$

2.2 Solving the quadratic

The Koide equation becomes

$$(\sqrt{3}+z)^2 = \tfrac{3}{2}(6+z^2),$$
$$3+2\sqrt{3}\,z+z^2 = 9+\tfrac{3}{2}z^2,$$
$$\tfrac{1}{2}z^2 - 2\sqrt{3}\,z + 6 = 0 \qquad \Longrightarrow \qquad z^2 - 4\sqrt{3}\,z + 12 = 0.$$

The discriminant is $\Delta = 48 - 48 = 0$: a **double root**,

$$\boxed{z = 2\sqrt{3}, \qquad z^2 = m_b = 12.}$$

2.3 Verification and Koide angle

$$Q = \frac{(\sqrt{3}+2\sqrt{3})^2}{6+12} = \frac{27}{18} = \frac{3}{2}. \checkmark$$

The scale parameter is $Q_0^{(2)} = (6+12)/3 = 6 = 3\,Q_0^{(1)}$. The Koide angle evaluates to $\delta_2 = 45^\circ = 3 \times 15^\circ$ in the same phase convention, confirming $\delta_{scb} = 3\,\delta_{0sc}$ at the exact chain point.

2.4 Mass ratios

$$m_s : m_c : m_b = \tfrac{(3-\sqrt{3})^2}{4} : \tfrac{(3+\sqrt{3})^2}{4} : 12 = (2-\sqrt{3}) : (2+\sqrt{3}) : 8.$$

3 Charges vs. masses: when does a Koide continuation exist?

3.1 The general discriminant

Given two charges a and b, we seek a third charge c satisfying $(a+b+c)^2 = \frac{3}{2}(a^2+b^2+c^2)$. This is a quadratic in c with discriminant

$$\boxed{\Delta = 12\big(a^2 + 4ab + b^2\big).}$$

3.2 Same-sign charges always continue

When a and b have the same sign, $4ab > 0$, so $\Delta > 0$: there are always two real solutions for c.

3.3 Opposite-sign charges have a forbidden band

When a and b have opposite signs, write $a/b = -r$ with $r > 0$. Then $\Delta = 12\,b^2(r^2 - 4r + 1)$. The roots of $r^2 - 4r + 1 = 0$ are $r = 2 \pm \sqrt{3}$, so:

- $\Delta > 0$ (two solutions) when $r < 2-\sqrt{3}$ or $r > 2+\sqrt{3}$;
- $\Delta = 0$ (double root) when $r = 2 \pm \sqrt{3}$;
- $\Delta < 0$ (no real solutions) when $2-\sqrt{3} < r < 2+\sqrt{3}$.

In terms of the mass ratio $m_1/m_2 = r^2$, the forbidden band is $(7-4\sqrt{3}) < m_1/m_2 < (7+4\sqrt{3})$.[1]

[1] Throughout this work, masses are obtained from charges by squaring, $m = q^2$, preserving holomorphy in q as a complex variable. An alternative construction using $m = |q|^2$ (i.e. $q\bar{q}$, where the mass arises from phase cancellation between a charge and its conjugate) would break holomorphy and is left for future investigation.

3.4 Masses vs. charges

Given two *masses*, one can always choose the same-sign pattern, which always yields two Koide continuations. **A pair of masses always has Koide descendants.** But a pair of *charges* with signs already fixed may not: the opposite-sign pattern fails in the forbidden band.

3.5 $\delta = 15^\circ$ sits exactly at the boundary

At $\delta = 15^\circ$, the charge ratio $q_2/q_3 = 2 - \sqrt{3}$ is *exactly* the lower boundary of the forbidden band. This is why Tuple 2 has a double root.

4 Perturbation: $\delta = \pi/12 + \varepsilon$

4.1 Perturbed charges of Tuple 1

To first order in ε:

$$q_1 = -\varepsilon + O(\varepsilon^2), \tag{3}$$

$$q_2 = \tfrac{3-\sqrt{3}}{2} + \tfrac{1+\sqrt{3}}{2}\,\varepsilon + O(\varepsilon^2), \tag{4}$$

$$q_3 = \tfrac{3+\sqrt{3}}{2} + \tfrac{1-\sqrt{3}}{2}\,\varepsilon + O(\varepsilon^2). \tag{5}$$

For $\varepsilon \neq 0$, the "up quark" acquires a small mass $m_u = \varepsilon^2 + O(\varepsilon^3)$.

4.2 Discriminant of Tuple 2

The charge ratio q_2/q_3 moves away from the critical value $2-\sqrt{3}$: $q_2/q_3 = (2-\sqrt{3})\Big(1 + \tfrac{4\sqrt{3}}{3}\,\varepsilon + O(\varepsilon^2)\Big)$. The discriminant of the Koide quadratic for Tuple 2 evaluates to

$$\Delta = 3\,\varepsilon\,(\varepsilon - 12) + O(\varepsilon^3).$$

- $\varepsilon = 0$: $\Delta = 0$, double root $z = 2\sqrt{3}$.
- $\varepsilon < 0$ (continuation-preserving side): $\Delta \approx 36\,|\varepsilon| > 0$, two solutions $z = 2\sqrt{3} \pm 6\sqrt{|\varepsilon|} + O(\varepsilon)$.
- $\varepsilon > 0$: $\Delta < 0$, no real continuation with the $(-,+)$ sign pattern.

This local bifurcation analysis concerns the exact opposite-sign continuation of Tuple 2. The measured positive-sign $(0, s, c)$ tuple discussed later sits slightly on the $\varepsilon > 0$ side of the critical point, whereas the continuation branches used in the inverse-tuple analysis are tracked from the $\varepsilon < 0$ side where the real opposite-sign continuation still exists.

5 Inverse-mass Koide tuples: $K(1/m_d,\, 1/m_s,\, 1/m_b)$

We now ask: given the known masses m_s and m_b from the chain, what values of m_d make the *inverse-mass* triplet $(1/m_d,\, 1/m_s,\, 1/m_b)$ satisfy the Koide equation?

Define the **inverse charges**

$$w_x \equiv \frac{1}{q_x} = \frac{1}{\sqrt{m_x}},$$

so that the "masses" in the inverse tuple are $w_x^2 = 1/m_x$. With our exact values:

$$w_s = \frac{1}{q_2} = \frac{2}{3-\sqrt{3}} = \frac{3+\sqrt{3}}{3}, \qquad w_b = \frac{1}{2\sqrt{3}} = \frac{\sqrt{3}}{6}.$$

5.1 Sign patterns

The charge ratio $w_s/w_b = q_b/q_s = 4\sqrt{3}/(3-\sqrt{3}) = 2(1+\sqrt{3}) \approx 5.46$, which exceeds $2+\sqrt{3} \approx 3.73$, so the opposite-sign pair $(-w_s,\, w_b)$ lies *outside* the forbidden band. Both sign patterns yield real solutions:

Case I: same-sign $(+w_s,\, +w_b,\, w_d)$.
Case II: opposite-sign $(-w_s,\, +w_b,\, w_d)$.

5.2 Case I: $(+w_s,\, +w_b,\, w_d)$

The quadratic in w_d is

$$w_d^2 - 2(2+\sqrt{3})\, w_d + \tfrac{3}{4} = 0,$$

with discriminant $\Delta_{\mathrm{I}} = 25 + 16\sqrt{3}$ (not a perfect square). The solutions are

$$w_d = (2+\sqrt{3}) \pm \tfrac{1}{2}\sqrt{25+16\sqrt{3}}\,,$$

giving $m_d = 1/w_d^2$.

5.3 Case II: $(-w_s,\, +w_b,\, w_d)$

The quadratic in w_d becomes

$$w_d^2 + \tfrac{2(6+\sqrt{3})}{3}\, w_d + \tfrac{25+16\sqrt{3}}{12} = 0.$$

The discriminant simplifies dramatically:

$$\boxed{\Delta_{\mathrm{II}} = 9, \qquad \sqrt{\Delta_{\mathrm{II}}} = 3.}$$

The two solutions are

$$w_d = -\frac{3+2\sqrt{3}}{6} \qquad \text{and} \qquad w_d = -\frac{21+2\sqrt{3}}{6}\,,$$

both negative (consistent with the opposite-sign pattern $(-w_s,\, +w_b,\, w_d)$ having $w_d < 0$).

5.3.1 Solution II-1: $w_d = -(3+2\sqrt{3})/6$

$$w_d^2 = \frac{(3+2\sqrt{3})^2}{36} = \frac{7+4\sqrt{3}}{12}, \qquad m_d = \frac{12}{7+4\sqrt{3}} = 12(7-4\sqrt{3}) = 12(2-\sqrt{3})^2.$$

The inverse-mass Koide angle of this solution is $\delta = 15^\circ$ **exactly**, matching Tuple 1.

5.3.2 Solution II-2: $w_d = -(21+2\sqrt{3})/6$

$$w_d^2 = \frac{(21+2\sqrt{3})^2}{36} = \frac{151+28\sqrt{3}}{12}, \qquad m_d = \frac{12}{151+28\sqrt{3}} = \frac{12(151-28\sqrt{3})}{143^2}.$$

5.4 Summary table

Define $W_0 = (w_d^2 + w_s^2 + w_b^2)/3$ for the inverse tuple and recall $Q_0 = Q_0^{(2)} = 6$ for the direct (s, c, b) tuple.
Solution II-1 is distinguished: its discriminant is a perfect square, its m_d is a clean radical $12(2-\sqrt{3})^2$, and its Koide angle is 15°—the same angle as Tuple 1. This self-referential property singles it out among the four solutions.

Case	w_d	$w_d^2 = \frac{1}{m_d}$	$m_d = \frac{1}{w_d^2}$	W_0	$\frac{Q_0}{W_0}$	δ_w
I-1	$(2+\sqrt{3}) + \frac{\sqrt{25+16\sqrt{3}}}{2}$ (7.362)	54.20	0.0184	18.92	0.317	9.85°
I-2	$(2+\sqrt{3}) - \frac{\sqrt{25+16\sqrt{3}}}{2}$ (0.102)	0.0104	96.36	0.861	6.97	6.68°
II-1	$-\dfrac{3+2\sqrt{3}}{6}$ (-1.077)	$\dfrac{7+4\sqrt{3}}{12}$ (1.161)	$12(2-\sqrt{3})^2$ (0.862)	$\dfrac{2+\sqrt{3}}{3}$ (1.244)	$18(2-\sqrt{3})$ (4.82)	**15.00°**
II-2	$-\dfrac{21+2\sqrt{3}}{6}$ (-4.077)	$\dfrac{151+28\sqrt{3}}{12}$ (16.62)	$\dfrac{12(151-28\sqrt{3})}{143^2}$ (0.060)	$\dfrac{14+3\sqrt{3}}{3}$ (6.399)	$\dfrac{18(14-3\sqrt{3})}{169}$ (0.94)	34.79°

Table 1: All Koide solutions for $K(1/m_d,\, 1/m_s,\, 1/m_b)$ at $\delta = 15°$. Cases I use $(+w_s, +w_b)$, Cases II use $(-w_s, +w_b)$. I-1 is the large-$|w_d|$ (small-m_d) branch; I-2 the small-$|w_d|$ (large-m_d) branch. Grey numbers are decimal evaluations of the exact expressions. Only solution II-1 has an angle in exact degrees. Units: $\sqrt{M_0} = 1$.

5.5 Galois duality: seed $(-1/q_s,\, +1/q_b)$ and its conjugate partner

The alternative seed pair $(+1/q_c,\, +1/q_b)$ is, up to an overall sign, the Galois conjugate of $(-w_s,\, w_b)$ under $\sqrt{3} \to -\sqrt{3}$ (which exchanges $q_s \leftrightarrow q_c$). Since $q_s q_c = 3$ and $q_b = 2\sqrt{3}$, the discriminant is again $\Delta = 9$ and the conjugate solutions are:

$$m_d^{(\pm)} = 12(7 \pm 4\sqrt{3}), \qquad m_d^{(\pm)} = \frac{12(151 \pm 28\sqrt{3})}{143^2}.$$

Each solution is the Galois conjugate of the corresponding one from Table **??**, with the product $m_d \cdot m_d^{(\mathrm{conj})} = 144$ (resp. $144/143^2$).

The solution $m_d = 12(7+4\sqrt{3})$ has Koide angle $\delta_w = 45°$ —echoing Tuple 2, just as our $m_d = 12(7-4\sqrt{3})$ echoes Tuple 1. The solution $m_d = 12(151+28\sqrt{3})/143^2$ has angle $\pi - \arctan(4\sqrt{3}-5) \approx 117.4°$ in a $k = 0, 1, 2$ input-ordered convention; in the Zenczykowski (magnitude-sorted) convention this becomes $\approx 2.6°$. This angle is algebraically exact in $\mathbb{Q}(\sqrt{3})$ but is *not* a rational multiple of π.

In total, both seed pairs produce exactly three angles that are multiples of $\pi/12$: the two forward-chain angles (15°, 45°) plus one seesaw angle that echoes one of them. The apparent abundance of "exact" angles in the k-ordered convention is partly an artifact: the phase depends on the k-assignment of the charges, and different assignments give different (but equally algebraic) values.

5.5.1 A complement near the lepton angle

The Zenczykowski angle of $m_d = 12(151+28\sqrt{3})/143^2$ (the Galois conjugate of our II-2) is

$$\delta_{d+} = \arctan\frac{7\sqrt{3}-10}{47} \approx 2.59°.$$

Its complement with respect to 15° has an exact tangent:

$$\tan(15° - \delta_{d+}) = \frac{5\sqrt{3}-8}{3} \approx 0.2201, \qquad 15° - \delta_{d+} = 12.41°.$$

This is close to the experimental lepton Koide angle $\delta_{e\mu\tau} = 12.73^\circ$ ($\tan\delta_{e\mu\tau} \approx 0.2260$), suggesting the decomposition

$$\delta_{d+} + \delta_{e\mu\tau} \approx 15^\circ.$$

At the exact $\delta = 15^\circ$ chain, the match is only approximate (deficit 0.32°); along the continuation-preserving branch with $\varepsilon < 0$, the gap could close.

6 Perturbation of the inverse-mass tuples

When $\delta = 15^\circ + \varepsilon$ with $\varepsilon < 0$, the double root $q_b = 2\sqrt{3}$ splits into two branches $q_{b\pm} = 2\sqrt{3} \pm 6\sqrt{|\varepsilon|} + O(\varepsilon)$. Each branch provides its own $w_{b\pm} = 1/q_{b\pm}$, and the inverse-mass Koide analysis of §5 must be repeated for each. This doubles the number of solutions from 4 to 8.

6.1 Exact projective II-1 identity along the deformed chain

Let q_b denote either real continuation root of the direct Koide triple $(-q_s, q_c, q_b)$. Its Koide condition is

$$q_b^2 - 4(q_c - q_s)\, q_b + (q_s^2 + 4q_sq_c + q_c^2) = 0.$$

The key observation is the projective identity

$$\left(-\frac{1}{q_s},\, \frac{1}{q_b},\, -\frac{q_c}{q_sq_b}\right) = -\frac{1}{q_sq_b}\,(q_b,\, -q_s,\, q_c).$$

Since the Koide condition is homogeneous of degree two and symmetric under permutations of the three entries, any overall rescaling or permutation of a Koide triple is again Koide. Therefore, once $(-q_s, q_c, q_b)$ is Koide,

$$\left(-\frac{1}{q_s},\, \frac{1}{q_b},\, -\frac{q_c}{q_sq_b}\right)$$

is automatically Koide as well.

For completeness, the same statement can be checked directly at the level of the inverse quadratic. Consider the opposite-sign inverse seed $(-1/q_s,\, +1/q_b,\, w_d)$. Its Koide polynomial is

$$w_d^2 + 4\left(\frac{1}{q_s} - \frac{1}{q_b}\right) w_d + \left(\frac{1}{q_s^2} + \frac{4}{q_sq_b} + \frac{1}{q_b^2}\right) = 0.$$

Substituting

$$w_d = -\frac{q_c}{q_sq_b}$$

$$\frac{q_b^2 - 4(q_c - q_s)\, q_b + q_s^2 + 4q_sq_c + q_c^2}{q_s^2q_b^2} = 0,$$

whose numerator is exactly the direct polynomial above. This direct substitution is redundant once the projective identity is noticed, but it gives a useful quadratic cross-check. The coincidence is therefore a projective/factorised identity, not an identity of the centred additive variables z_0, z_i, w_0, ω_i.

Among the six permutations of $(-q_s, q_c, q_b)$, this is the only generic direct/inverse proportionality with fixed second inverse entry $+1/q_b$: the others collapse only on special loci such as $q_b = q_c$ or $q_b^2 = q_sq_c$. By continuity from the exact $\varepsilon = 0$ point, the matched opposite-sign inverse root is precisely the II-1 branch on both b_+ and b_- continuations.

One isolated special locus is worth recording because it fixes an angle. On the lower direct branch, imposing $q_{b-} = q_c$ in the direct quadratic gives

$$-2q_c^2 + 8q_sq_c + q_s^2 = 0 \qquad \Longrightarrow \qquad \frac{q_c}{q_s} = 2 + \frac{3\sqrt{2}}{2}.$$

Numerically this occurs at

$$\varepsilon_{\rm eq} \approx -0.04158609, \qquad \delta_{\rm eq} \approx 12.61729^\circ.$$

This does not create a second generic direct/inverse coincidence family, but it marks a distinguished angle at which two direct entries become equal and the existing II-1 permutation becomes partially degenerate.

6.2 Numerical example: $\varepsilon = -0.001$

At $\varepsilon = -0.001$ (i.e. $\delta \approx 14.94^\circ$), the perturbed Tuple 1 charges are $q_1 = 0.001$, $q_2 = 0.6326$, $q_3 = 2.3664$, and the b charge splits into $q_{b+} = 3.6573$ and $q_{b-} = 3.2778$.

Case	Branch	w_d	w_d^2	$m_d = 1/w_d^2$	W_0	δ_w
I-1	b_+	$+7.301$	53.30	0.019	18.63	10.07°
I-1	b_-	$+7.454$	55.57	0.018	19.39	9.63°
I-2	b_+	$+0.116$	0.013	74.71	0.862	5.63°
I-2	b_-	$+0.089$	0.008	126.49	0.867	7.70°
II-1	b_+	-1.023	1.046	0.956	1.207	17.05°
II-1	b_-	-1.141	1.302	0.768	1.298	12.87°
II-2	b_+	-4.207	17.70	0.057	6.756	35.68°
II-2	b_-	-3.961	15.69	0.064	6.095	33.83°

Table 2: All eight inverse-mass Koide solutions at $\varepsilon = -0.001$. Each solution from Table **??** forks into a b_+ and b_- branch.

6.3 Observations

- Each of the four $\varepsilon = 0$ solutions splits smoothly into a pair, one from each q_b branch.
- The Cases I discriminant $\Delta_{\rm I} = 25 + 16\sqrt{3} + O(\sqrt{|\varepsilon|})$ remains positive; Cases II discriminant perturbs from 9 but stays positive.
- The distinguished solution II-1 ($\delta_w = 15^\circ$ at $\varepsilon = 0$) splits into $\delta_w = 17.05^\circ$ (b_+) and $\delta_w = 12.87^\circ$ (b_-). The b_- branch is notable: at the experimental lepton angle $\delta_{e\mu\tau} \approx 12.73^\circ$ ($\varepsilon \approx -0.040$), this branch would approach a value near the physical lepton angle itself.

7 Comparison with experimental masses

We now confront the algebraic results of §§1–6 with the experimental quark and lepton masses.

7.1 Reference angles for the standard Koide chain

Using PDG 2024 $\overline{\rm MS}$ masses ($m_d = 4.67$ MeV, $m_s = 93.4$ MeV, $m_c = 1.27$ GeV, $m_b = 4.18$ GeV at $\mu = 2$ GeV or $m_q(m_q)$; $m_t = 172.69$ GeV pole; m_e, m_μ, m_τ pole):

Tuple	Signs	δ	Q	Note
Standard Koide chain				
e, μ, τ (pole)	$(+,+,+)$	$12.73°$	1.50001	exact Koide
(u, s, c)	$(+,+,+)$	$13.28°$	1.602	
$(0, s, c)$ $[m_u \to 0]$	$(+,+,+)$	$15.20°$	1.505	near Koide
(s, c, b)	$(-,+,+)$	$37.21°$	1.482	$\approx 3\delta_{e\mu\tau}$
(c, b, t)	$(+,+,+)$	$3.93°$	1.494	near Koide
Other quark triples				
(d, s, b)	$(+,+,+)$	$6.31°$	1.367	
(d, s, c)	$(+,+,+)$	$12.33°$	1.647	$\approx \delta_{e\mu\tau}$
(u, c, t)	$(+,+,+)$	$4.26°$	1.178	
Inverse-mass tuples				
$K(1/d, 1/s, 1/b)$	$(+,+,+)$	$10.70°$	1.503	near Koide
$K(1/e, 1/\mu, 1/\tau)$	$(+,+,+)$	$2.73°$	1.174	$\approx \delta_{d+}$
$K(1/u, 1/c, 1/t)$	$(+,+,+)$	$1.91°$	1.090	

Table 3: Koide angles and Q-parameters for chain tuples, other quark triples, and inverse-mass tuples (PDG 2024, $m_q(m_q)$ convention). Only $K(1/d, 1/s, 1/b)$ among inverse-mass tuples approaches $Q = 3/2$.

Observations on Table ??

- The $(0, s, c)$ tuple at $\delta = 15.20°$ confirms that the idealised $\delta = 15°$ of §1 is a good approximation. The $(-s, c, b)$ angle of $37.21°$ is close to $3 \times 12.73° = 38.2°$, consistent with the factor-of-three relation $\delta_{scb} \approx 3\,\delta_{e\mu\tau}$.
- Among all inverse-mass tuples, **only** $K(1/d, 1/s, 1/b)$ **approaches Koide**: $Q = 1.503$ (0.2% from 3/2). The inverse lepton tuple $K(1/e, 1/\mu, 1/\tau)$ gives $Q = 1.17$ and the inverse up-type $K(1/u, 1/c, 1/t)$ gives $Q = 1.09$—both far from 3/2. This singles out the down-type quarks.
- The inverse lepton angle $\delta(K(1/e, 1/\mu, 1/\tau)) = 2.73°$ is remarkably close to the Galois conjugate seesaw angle $\delta_{d+} = 2.59°$ from §**??**. While the inverse lepton tuple does *not* satisfy Koide ($Q = 1.17$), the near-coincidence of angles is intriguing.
- The ratios $\delta_w/\delta_{e\mu\tau} = 10.70/12.73 = 0.840$ and $\delta_{e\mu\tau}/\delta_{0sc} = 12.73/15.20 = 0.837$ are nearly identical, suggesting a geometric progression $\delta_w \approx \delta^2_{e\mu\tau}/\delta_{0sc}$.
- The (d, s, c) tuple has $\delta = 12.33°$, close to $\delta_{e\mu\tau} = 12.73°$. This is natural: in the exact chain, the $(0, s, c)$ tuple has $\delta = 15°$, and replacing the zero by $m_d > 0$ pushes the angle downward. The (d, s, c) tuple is the "wrong light quark" version of (u, s, c) ($\delta = 13.28°$); both sit below $15°$ and bracket the lepton angle. The proximity of (d, s, c) to $\delta_{e\mu\tau}$ may reflect the same mechanism that links quark and lepton Koide angles in the chain.

7.2 The inverse-mass tuple $K(1/m_d,\, 1/m_s,\, 1/m_b)$

With all positive signs and PDG central values in the $m_q(m_q)$ convention:

$$Q = 1.503, \qquad \delta_w = 10.70°.$$

The deviation from Koide, $|Q - 3/2| = 3.3 \times 10^{-3}$, is larger than the lepton Koide (1.4×10^{-5}) but much smaller than the direct (d, s, b) tuple ($Q = 1.37$).

To achieve $Q = 3/2$ exactly, one needs $m_d = 4.61$ MeV, a -1.4% shift from the PDG central value 4.67 ± 0.48 MeV—well within the 10% experimental uncertainty.

7.3 RG running

At one-loop QCD, all quark masses of the same charge run with a common anomalous dimension, so their ratios—and hence the Koide angle—are RG invariants within each flavour-number regime. Threshold corrections at $\mu = m_b$ and $\mu = m_t$ introduce small shifts.

Convention	m_d (MeV)	m_s (MeV)	m_c (GeV)	m_b (GeV)
Common $\mu = 2$ GeV	4.67	93.4	1.167	4.667
$m_q(m_q)$	4.67	93.4	1.270	4.180

Convention	δ_{0sc}	Q_{0sc}	δ_{-scb}	Q_{-scb}	δ_w	Q_w
Common $\mu = 2$ GeV	15.93°	1.524	34.09°	1.453	10.79°	1.4992
$m_q(m_q)$	15.20°	1.505	37.21°	1.482	10.70°	1.5033

Table 4: Koide angles for the chain tuples and the inverse-mass tuple $K(1/d, 1/s, 1/b)$ in two mass conventions. At 1-loop QCD the mass anomalous dimension is flavour-universal, so all angles are exactly scale-invariant within each convention—only the choice of convention matters, not the RG scale. The $\delta_{-scb}/\delta_{0sc}$ ratio is 2.14 at common scale vs. 2.45 in $m_q(m_q)$.

Observations

- **$(0, s, c)$ is above 15° in both conventions.** $\delta_{0sc} = 15.93°$ (common scale) or 15.20° ($m_q(m_q)$). In both cases $\varepsilon > 0$ in the language of §4, placing us past the bifurcation where the exact opposite-sign continuation has no real solutions. The experimental $(-s, c, b)$ tuple has $Q = 1.45$–1.48, close to but not equal to 3/2—consistent with being near but past the critical angle.

- **1-loop QCD preserves all angles trivially.** The mass anomalous dimension is flavour-universal at one loop, so all quark mass ratios are exactly preserved under running. The angles in Table ?? are strictly scale-invariant; only the choice of mass convention (common scale vs. $m_q(m_q)$) matters. Yukawa contributions to the running, which are flavour-dependent (dominated by the top Yukawa), would break this invariance—particularly for m_b. The question of how much δ_w and δ_{-scb} actually run under full SM RGEs is the natural next step for this programme.

- **Q_w is closest to 3/2 at common scale.** $Q_w = 1.4992$ vs. 1.5033 in $m_q(m_q)$. The exact-Koide prediction at $\mu = 2$ GeV is $m_d = 4.69$ MeV (+0.4% from PDG). With pole $m_b = 4.78$ GeV, $Q_w = 1.4983$.

8 Which branch are we on?

The exact chain at $\delta = 15°$ has a double root for m_b and four seesaw solutions for m_d. At the physical point, δ_{0sc} from Table ?? lies slightly above 15°, i.e. on the $\varepsilon > 0$ side of the bifurcation. For branch selection, however, we follow the real continuation-preserving side $\varepsilon < 0$, where the double root splits into $q_{b\pm}$ and each seesaw solution forks into two branches, giving eight candidates. We now ask: for which branches does the inverse-mass Koide angle δ_w match the experimental value $\approx 10.70°$, and does the rest of the chain produce sensible mass ratios?

8.1 Matching $\delta_w = 10.70°$

Here δ_w is the Zenczykowski angle of the *mass triple* (m_d, m_s, m_b) generated by a given signed seed, with the sign pattern recorded separately in the branch label. The observable inverse-mass tuple $K(1/d, 1/s, 1/b)$ uses all positive entries; the signed seeds are an auxiliary device for selecting which exact Koide continuation produces that positive mass spectrum. The comparison is therefore between two descriptions of the same masses, not between two incompatible observables.

A numerical scan over $\varepsilon \in (-0.15, 0)$ finds exactly three branches where δ_w passes through $10.70°$:

Branch	ε (rad)	δ_1	m_d/m_s	m_s/m_c	m_b/m_c	δ_{-scb}	m_d (MeV)	m_b (MeV)
I-1/b_+	$\mathbf{-0.0137}$	**14.2°**	**0.050**	**0.067**	**3.16**	**37.8°**	**4.3**	**4033**
I-2/b_-	-0.0169	14.0°	944	0.066	1.34	54.2°	80021	1707
II-1/b_-	-0.1101	8.7°	0.58	0.041	0.58	49.3°	31	763

Table 5: Branches matching $\delta_w \approx 10.70°$. Physical masses obtained by scaling $Z_0 = m_s + m_c$ to 1363 MeV. Experimental ratios: $m_d/m_s \approx 0.050$, $m_b/m_c \approx 3.3$.

8.2 Filtering by mass ratios

- **I-2/b_-: excluded.** $m_d/m_s = 944$: the down quark is heavier than the bottom.
- **II-1/b_-: excluded.** $m_b/m_c = 0.58$: the bottom is lighter than charm.
- **I-1/b_+: viable.** All mass ratios are close to experiment:
  - $m_d/m_s = 0.050$ (exp: 0.050),
  - $m_s/m_c = 0.067$ (exp: 0.074),
  - $m_b/m_c = 3.16$ (exp: 3.29),
  - $\delta_{-scb} = 37.8°$ (vs. $3\delta_{e\mu\tau} = 38.2°$).

8.3 The surviving solution

The unique viable branch is **I-1/b_+**: the same-sign seesaw ($+w_s$, $+w_b$, w_d) with the heavy b-branch, at $\varepsilon = -0.0137$ ($\delta_1 = 14.2°$).

At this point the chain predicts (scaling to $Z_0 = 1363$ MeV):

$$m_u = 0.04 \text{ MeV}, \quad m_d = 4.3 \text{ MeV}, \quad m_s = 86 \text{ MeV}, \quad m_c = 1277 \text{ MeV}, \quad m_b = 4033 \text{ MeV}.$$

All except the very small m_u lie within about 1–2σ of PDG central values; m_u comes out parametrically too light in this fit. The $(-s, c, b)$ angle 37.8° is within 1% of $3\delta_{e\mu\tau} = 38.2°$, preserving the Zenczykowski relation.

Note that $\delta_1 = 14.2°$ is the angle of the *first* tuple $(0, s, c)$ at this ε — intermediate between the idealised 15° and the experimental lepton angle 12.73°. The seesaw lives on the same-sign, b_+ branch: the only combination that simultaneously satisfies the inverse-mass Koide equation and produces a realistic quark spectrum.

8.4 Near-compatibility of two constraints

The system has one free parameter (ε) and two natural constraints:

(A) $\delta_w = 10.70°$ (experimental inverse-mass angle),

(B) $\delta_{-scb} = 3\,\delta_{e\mu\tau} = 38.20°$ (Zenczykowski relation).

Fixing (A) gives $\varepsilon_A = -0.0137$, $\delta_{-scb} = 37.79°$ (miss $-0.41°$). Fixing (B) gives $\varepsilon_B = -0.0121$, $\delta_w = 10.65°$ (miss $-0.05°$). The gap $|\varepsilon_A - \varepsilon_B| = 0.0016$ rad ($0.09°$) is small: the two constraints are nearly compatible, with a residual of $\sim 1\%$ in δ_{-scb}. This residual could plausibly be absorbed by Yukawa corrections to m_b running, which we have not included.

9 Other tuples

9.1 The chiral tuple $(m_d,\, 0,\, m_s)$

Descending the chain from $(0,\, m_s,\, m_c)$ with $m_u = 0$, the previous link is $(m_d,\, 0,\, m_s)$. The Koide equation gives $q_d = (2 \pm \sqrt{3})\, q_s$, i.e.

$$q_{d+} = (2+\sqrt{3})\, q_s = \frac{3+\sqrt{3}}{2} = q_c, \qquad q_{d-} = (2-\sqrt{3})\, q_s = \frac{9-5\sqrt{3}}{2}.$$

The "+" solution gives back the charm charge—the chain *loops*: descending from $(0, s, c)$ recovers m_c as one branch. The physical solution is q_{d-}, with

$$m_{d-} = (7-4\sqrt{3})\, m_s, \qquad M_0^{(d,0,s)} = (7-4\sqrt{3})\, M_0^{(0,s,c)}.$$

Both solutions have $\delta = 15°$ exactly (a zero entry always reproduces the seed angle). The scale parameter M_0 is multiplied by $(2-\sqrt{3})^2 = 7-4\sqrt{3}$—the same factor that relates consecutive Koide tuples in the chain.

Replacing m_d by the *chiral mass* $m_u + m_d$ (the quantity entering the chiral condensate) gives a physical near-Koide tuple:

$$(m_u{+}m_d,\ 0,\ m_s) = (6.83,\ 0,\ 93.4)\ \text{MeV}: \qquad Q = 1.504, \quad \delta = 15.15°.$$

The chiral combination works better than m_d alone (which gives $Q = 1.43$, $\delta = 12.3°$): it sits just above $15°$, consistent with the bifurcation analysis of §4.

9.2 The $(s,\, b,\, t)$ tuple

The top charge from (c, b, t) at $\delta = 15°$ simplifies via $\sqrt{4+2\sqrt{3}} = 1+\sqrt{3}$:

$$q_t = \frac{15+19\sqrt{3}}{2}, \qquad m_t = \frac{654+285\sqrt{3}}{2}.$$

The key mass ratios in the exact chain are:

$$\frac{m_t}{m_c} = 151 - 28\sqrt{3}, \qquad \frac{m_b}{m_c} = 16 - 8\sqrt{3} = 8(2-\sqrt{3}), \qquad \frac{m_c}{m_s} = 7+4\sqrt{3}.$$

Note that 151 and 28 are the same integers appearing in the inverse-mass seesaw solution $m_d^{(\text{II-2})} = 12(151-28\sqrt{3})/143^2$.

The (s, b, t) tuple gives

$$Q_{sbt} = 1.342, \qquad \delta_{sbt} = 6.38^\circ \qquad \text{(exact chain)},$$

far from 3/2. With PDG masses: $Q = 1.36$, $\delta = 7.2^\circ$. Neither sign pattern helps ($Q_{-s,b,t} = 1.22$). This tuple is genuinely non-Koide at every scale.

9.3 The corrected inverse tuple $K(1/s,\ 1/c,\ \pm 1/t)$

The tuple $K(1/s,\ 1/c,\ 0)$ is the inversion of $(0,\ s,\ c)$ and has $\delta = 15^\circ$, $Q = 3/2$ exactly. The inverse top charge is

$$\frac{1}{q_t} = \frac{19\sqrt{3} - 15}{429}, \qquad 429 = 3 \times 11 \times 13.$$

Adding $\pm 1/q_t$ as a correction to the zero entry:

Tuple	δ (exact)	Q (exact)	δ (PDG)	Q (PDG)
$K(1/s,\ 1/c,\ 0)$	15.00°	1.500	15.20°	1.505
$K(1/s,\ 1/c,\ +1/t)$	13.78°	1.562	14.13°	1.560
$K(1/s,\ 1/c,\ -1/t)$	16.17°	1.437	16.23°	1.450

Adding $+1/q_t$ pushes δ *below* 15° (toward the lepton angle); adding $-1/q_t$ pushes it above. Neither achieves $Q = 3/2$, but the (+) correction brings δ closer to $\delta_{e\mu\tau} = 12.73^\circ$ (13.78° at the chain point, 14.13° with PDG masses). The correction is small ($\sim 1^\circ$) because $1/q_t = 0.042 \ll 1/q_c = 0.423$—the top mass suppresses the third entry by an order of magnitude.

10 Meson Koide tuples

The Koide equation can be applied to meson masses. We scan all triples of pseudoscalar mesons with both sign patterns $(+, +, +)$ and $(-, +, +)$.

10.1 Best pseudoscalar triples

Tuple	Signs	δ	Q	$\lvert Q-3/2\rvert$
Type $(-\pi,\ D_s,\ X)$				
$(\pi^0,\ D_s,\ \eta_c)$	$(-, +, +)$	51.73°	1.5006	6×10^{-4}
$(\pi^\pm,\ D_s,\ B^\pm)$	$(-, +, +)$	40.79°	1.4984	1.6×10^{-3}
$(\pi^\pm,\ D_s,\ B_s)$	$(-, +, +)$	40.49°	1.4978	2.2×10^{-3}
$(\pi^0,\ D_s,\ B_c)$	$(-, +, +)$	37.65°	1.4962	3.8×10^{-3}
Type $(-\pi,\ D,\ B)$				
$(\pi^0,\ D^\pm,\ B^\pm)$	$(-, +, +)$	39.89°	1.4929	7.1×10^{-3}
$(\pi^\pm,\ D^\pm,\ B^\pm)$	$(-, +, +)$	39.94°	1.4864	1.4×10^{-2}
Other				
$(\pi^\pm,\ \pi^0,\ \eta_b)$	$(+, +, +)$	0.11°	1.4981	1.9×10^{-3}

Table 6: Best pseudoscalar meson Koide tuples (PDG 2024 masses). The $(-\pi, D_s, \eta_c)$ tuple rivals the lepton Koide in precision. All require a sign flip on the pion.

The $(-\pi^0,\, D_s,\, \eta_c)$ tuple is remarkable: $|Q - 3/2| = 6 \times 10^{-4}$, only $40\times$ worse than the lepton Koide. The pattern is consistent: *all* good meson triples involve a sign flip on the pion (the lightest member) and pair it with a charmed and a heavier meson. The D_s (containing an s quark) works better than the D, suggesting the strange content improves the match.

10.2 Vector mesons

No triple of vector mesons $(\rho, \omega, K^*, \phi, D^*, B^*, J/\psi, \Upsilon)$ comes close to Koide: the best is $(-\rho,\, B_s^*,\, \Upsilon)$ with $Q = 1.31$. Vector mesons do not form Koide tuples.

10.3 Inverse-mass mesons

Among pure meson triples, no inverse-mass tuple approaches Koide: the best is $K(1/\pi^0,\, 1/B_c,\, 1/\eta_b)$ with $Q = 1.55$.

10.4 Including W, Z, and H

Adding the electroweak bosons ($M_W = 80.36$, $M_Z = 91.19$, $M_H = 125.2$ GeV) to the scan produces several near-Koide mixed meson-boson tuples:

Tuple	Signs	δ	Q	$\lvert Q-3/2\rvert$
$K(1/\pi^0,\, 1/\eta_c,\, 1/W)$	$(+,+,+)$	$9.66°$	1.5013	1.3×10^{-3}
$(\pi^0,\, B_c,\, H)$	$(+,+,+)$	$10.75°$	1.5024	2.4×10^{-3}
$(D^\pm,\, D^0,\, H)$	$(+,+,+)$	$0.01°$	1.5033	3.3×10^{-3}
$(\eta',\, D^0,\, Z)$	$(+,+,+)$	$2.29°$	1.5047	4.7×10^{-3}
$(-B_c,\, Z,\, H)$	$(-,+,+)$	$53.70°$	1.4931	6.9×10^{-3}

The pure boson triple (W, Z, H) gives $Q = 2.97$—not Koide. But mixed meson-boson tuples can achieve $|Q-3/2| \sim 10^{-3}$, comparable to $K(1/d, 1/s, 1/b)$. The inverse-mass tuple $K(1/\pi^0, 1/\eta_c, 1/W)$ is particularly clean: all positive signs, $\delta = 9.7°$, and Q within 0.1% of 3/2. Whether these reflect structure or coincidence across the meson-boson mass landscape requires further investigation.

10.5 Baryons: direct and inverse

Using PDG 2025 baryon masses, we scan all triples among the 22 ground-state baryons (p, n, Λ, $\Sigma^{+,0,-}$, $\Xi^{0,-}$, Ω^-, Λ_c^+, $\Sigma_c^{++,+,0}$, $\Xi_c^{+,0}$, Ω_c^0, Λ_b^0, $\Sigma_b^{+,-}$, $\Xi_b^{0,-}$, Ω_b^-).

Direct: The best triple is $(-p,\, \Sigma_b^-,\, \Omega_b^-)$ with $Q = 1.19$—far from 3/2. No baryon triple with any sign pattern approaches Koide.

Inverse: The best is $K(1/p,\, 1/\Sigma_b^-,\, 1/\Omega_b^-)$ with $Q = 2.45$—even farther.

The Koide property appears specific to: charged leptons (exact), the quark chain $\{(0sc), (-scb), (cbt)\}$ and its inverse-mass descendant $K(1/d, 1/s, 1/b)$ (approximate), and certain pseudoscalar meson triples with sign-flipped pion (approximate). Baryons do not participate.

11 Preon charge patterns

For any signed direct tuple with charges $q_i = s_i\sqrt{m_i}$, one can extract a common piece z_0 and a traceless part z_i by

$$q_i = z_0 + z_i, \qquad z_0 = \frac{q_1 + q_2 + q_3}{3}, \qquad z_1 + z_2 + z_3 = 0.$$

Likewise, for any inverse tuple with signed inverse charges $w_i = s_i/\sqrt{m_i}$,

$$w_i = w_0 + \omega_i, \qquad w_0 = \frac{w_1 + w_2 + w_3}{3}, \qquad \omega_1 + \omega_2 + \omega_3 = 0.$$

The observed masses are recovered as

$$m_i = (z_0 + z_i)^2, \qquad \frac{1}{m_i} = (w_0 + \omega_i)^2.$$

If one wishes to interpret the constituents themselves as "mean preon masses", the corresponding scales are obtained by squaring the direct charges or inverse-squaring the inverse charges: z_0^2, z_i^2 and $1/w_0^2, 1/\omega_i^2$.

It is also convenient to define the centred ratios

$$R_z \equiv \frac{z_1^2 + z_2^2 + z_3^2}{3z_0^2}, \qquad R_w \equiv \frac{\omega_1^2 + \omega_2^2 + \omega_3^2}{3w_0^2}.$$

Since $Q = 9z_0^2/(3z_0^2 + z_1^2 + z_2^2 + z_3^2)$ (and similarly for w), one has

$$R = \frac{3}{Q} - 1.$$

Thus exact Koide corresponds simply to $R = 1$, while near-Koide tuples have R close to 1.

11.1 Direct tuples

For the two exact quark-chain anchors, the decomposition is especially simple:

$$(0, s, c): \qquad z_0 = 1, \qquad (z_1, z_2, z_3) = \left(-1, \frac{1-\sqrt{3}}{2}, \frac{1+\sqrt{3}}{2}\right),$$

$$(-s, c, b): \qquad z_0 = \sqrt{3}, \qquad (z_1, z_2, z_3) = \left(-\frac{3+\sqrt{3}}{2}, \frac{3-\sqrt{3}}{2}, \sqrt{3}\right).$$

It is also useful to record an alternative *factorised* normalization for the exact tuples. Writing $q_i = \bar{z}\,\hat{z}_i$ with $\bar{z} = (q_1 + q_2 + q_3)/3$ and $\hat{z}_1 + \hat{z}_2 + \hat{z}_3 = 3$, and defining

$$a \equiv \frac{3-\sqrt{3}}{2}, \qquad b \equiv \frac{3+\sqrt{3}}{2},$$

one has the exact reduced shapes

$$(0, s, c) = 1 \cdot (0, a, b), \qquad (d, 0, s)_{\text{phys}} = (2 - \sqrt{3}) \cdot (a, 0, b), \qquad (-s, c, b) = \sqrt{3} \cdot (a - 1, b - 1, 2).$$

This factorised notation is not the centred preon decomposition used in the tables below, but it makes the exact shape relations transparent.

The same centred decomposition exists for the other exact and empirical tuples:

The direct patterns suggest a clear hierarchy. The lepton tuple and the exact/fitted quark-chain tuples all sit very close to $R_z = 1$. The empirical $(0, s, c)$ and $(-s, c, b)$ tuples remain near this surface, whereas (d, s, b), (u, c, t) and (s, b, t) drift farther away. A simple scan over the centred charges and their preon masses shows that the quark chain is unusually rigid. In the exact solutions, passing from $(0, s, c)$ to $(-s, c, b)$ multiplies each centred preon mass by a factor of 3. Empirically the pattern softens but survives in recognisable form: the common piece nearly doubles, $z_0 : 15.10 \to 30.21$, so its preon mass nearly quadruples, $z_0^2 : 228 \to 913$ MeV. Few other families exhibit such a clean component-wise scaling.

Tuple	Signs	preon masses	z_0	(z_1, z_2, z_3)	R_z
Exact tuples, $\sqrt{M_0} = 1$ so masses are unitless					
exact $(0, s, c)$	$(+, +, +)$	$(1.000; 1.000, 0.134, 1.866)$	1.000	$(-1.000, -0.366, 1.366)$	1.000
exact $(-s, c, b)$	$(-, +, +)$	$(3.000; 5.598, 0.402, 3.000)$	1.732	$(-2.366, 0.634, 1.732)$	1.000
exact $(d, 0, s)$ [phys.]	$(+, +, +)$	$(0.072; 0.010, 0.072, 0.134)$	0.268	$(-0.098, -0.268, 0.366)$	1.000
exact (c, b, t)	$(+, +, +)$	$(98.57; 57.19, 41.78, 196.7)$	9.928	$(-7.562, -6.464, 14.026)$	1.000
exact (s, b, t)	$(+, +, +)$	$(87.44; 75.98, 34.65, 213.3)$	9.351	$(-8.717, -5.887, 14.604)$	1.235
Empirical / fitted tuples, masses in MeV					
fit $(-s, c, b)$ [I-$1/b_+$]	$(-, +, +)$	$(899.3; 1541.6, 33.02, 1123.4)$	29.989	$(-39.263, 5.746, 33.517)$	1.000
chiral $(u + d, 0, s)$	$(+, +, +)$	$(16.75; 2.188, 16.75, 31.05)$	4.093	$(-1.479, -4.093, 5.572)$	0.995
(e, μ, τ)	$(+, +, +)$	$(313.8; 289.0, 55.30, 597.2)$	17.716	$(-17.001, -7.437, 24.437)$	1.000
(u, s, c)	$(+, +, +)$	$(243.1; 199.4, 35.12, 401.9)$	15.590	$(-14.121, -5.926, 20.047)$	0.873
$(0, s, c)$	$(+, +, +)$	$(228.0; 228.0, 29.55, 421.8)$	15.100	$(-15.100, -5.436, 20.537)$	0.993
$(-s, c, b)$	$(-, +, +)$	$(912.6; 1589.8, 29.47, 1186.4)$	30.209	$(-39.873, 5.429, 34.444)$	1.025
(d, s, c)	$(+, +, +)$	$(250.3; 186.6, 37.90, 392.7)$	15.821	$(-13.660, -6.156, 19.816)$	0.822
(d, s, b)	$(+, +, +)$	$(649.9; 544.4, 250.5, 1533.5)$	25.493	$(-23.332, -15.828, 39.160)$	1.194
(c, b, t)	$(+, +, +)$	$(29566.8; 18581.2, 11512.7, 59345.8)$	171.9	$(-136.3, -107.3, 243.6)$	1.008
(u, c, t)	$(+, +, +)$	$(22767.5; 22326.1, 13283.0, 70050.7)$	150.9	$(-149.4, -115.3, 264.7)$	1.547
PDG (s, b, t)	$(+, +, +)$	$(26664.4; 23601.6, 9729.7, 63638.9)$	163.3	$(-153.6, -98.639, 252.3)$	1.212

Table 7: Direct charge decompositions $q_i = z_0 + z_i$ for the quark/lepton tuples discussed in the text. The mass column lists the preon component masses $(z_0^2; z_1^2, z_2^2, z_3^2)$. Exact Koide corresponds to $R_z = 1$. Empirical rows use the same PDG conventions as Table **??**.

11.2 Inverse tuples

For inverse tuples there are two natural conventions. One may either decompose the observable positive inverse-mass triple $K(1/m_1, 1/m_2, 1/m_3)$, or keep the signed inverse charges that define a specific seesaw branch. To preserve the branch information, the exact I/II solutions below are written with their signed inverse charges in the (d, s, b) order.

Tuple / seed	Signs	preon masses	w_0	$(\omega_1, \omega_2, \omega_3)$	R_w
Exact inverse solutions, $\sqrt{M_0} = 1$ so preon masses are unitless					
I-1	$(+, +, +)$	$(0.106; 0.054, 0.445, 0.129)$	3.076	$(4.286, -1.499, -2.787)$	1.000
I-2	$(+, +, +)$	$(2.324; 3.257, 1.178, 7.413)$	0.656	$(-0.554, 0.921, -0.367)$	1.000
II-1	$(-, -, +)$	$(1.608; 12.000, 1.608, 0.862)$	-0.789	$(-0.289, -0.789, 1.077)$	1.000
II-2	$(-, -, +)$	$(0.313; 0.191, 22.392, 0.232)$	-1.789	$(-2.289, 0.211, 2.077)$	1.000
exact $K(1/s, 1/c, 0)$	$(+, +, 0)$	$(2.250; 1.206, 16.794, 2.250)$	0.667	$(0.911, -0.244, -0.667)$	1.000
exact $K(1/s, 1/c, +1/t)$	$(+, +, +)$	$(2.159; 1.244, 15.031, 2.450)$	0.681	$(0.897, -0.258, -0.639)$	0.920
exact $K(1/s, 1/c, -1/t)$	$(+, +, -)$	$(2.347; 1.170, 18.887, 2.073)$	0.653	$(0.925, -0.230, -0.694)$	1.088
Empirical / fitted inverse tuples, preon masses in MeV					
PDG $K(1/d, 1/s, 1/b)$	$(+, +, +)$	$(26.60; 13.84, 122.3, 31.41)$	0.194	$(0.269, -0.090, -0.178)$	0.996
PDG $K(1/e, 1/\mu, 1/\tau)$	$(+, +, +)$	$(3.896; 1.256, 5.968, 4.288)$	0.507	$(0.892, -0.409, -0.483)$	1.554
PDG $K(1/u, 1/c, 1/t)$	$(+, +, +)$	$(17.81; 5.085, 22.915, 18.177)$	0.237	$(0.443, -0.209, -0.235)$	1.753
PDG $K(1/s, 1/c, 0)$	$(+, +, 0)$	$(520.2; 281.3, 4014.0, 520.2)$	0.044	$(0.060, -0.016, -0.044)$	0.993
PDG $K(1/s, 1/c, +1/t)$	$(+, +, +)$	$(501.7; 289.0, 3635.1, 560.5)$	0.045	$(0.059, -0.017, -0.042)$	0.923
PDG $K(1/s, 1/c, -1/t)$	$(+, +, -)$	$(539.8; 273.8, 4455.3, 484.1)$	0.043	$(0.060, -0.015, -0.045)$	1.069
fit $K(1/d, 1/s, 1/b)$ [I-$1/b_+$]	$(+, +, +)$	$(24.52; 12.73, 112.9, 28.85)$	0.202	$(0.280, -0.094, -0.186)$	0.998

Table 8: Inverse charge decompositions $w_i = w_0 + \omega_i$. The mass column lists the inverse preon-component masses $(1/w_0^2; 1/\omega_1^2, 1/\omega_2^2, 1/\omega_3^2)$. The exact branch solutions use the signed inverse charges that generate the seesaw branches; the empirical inverse tuples use the observable positive inverse masses.

The distinguished inverse down-type solutions again sit near $R_w = 1$: exact II-1 gives $R_w = 1$ exactly, the empirical $K(1/d, 1/s, 1/b)$ gives $R_w = 0.996$, and the fitted surviving branch gives $R_w = 0.998$. By contrast, the inverse lepton and inverse up-type tuples move far away from this centred surface. In the corresponding factorised notation $w_i = \bar{w}\,\hat{w}_i$, the stronger statement proved in §**??** is that the continued II-1 branch is projectively identical to the direct continued branch for both $q_{b\pm}$:

$$\left(-\frac{1}{q_s},\ \frac{1}{q_b},\ -\frac{q_c}{q_s q_b}\right) = -\frac{1}{q_s q_b}\,(q_b,\ -q_s,\ q_c).$$

At the exact $\delta = 15°$ point this reduces to the simple radical shape

$$(-w_s,\, +w_b,\, w_d)_{\mathrm{II}-1} = -\frac{3+\sqrt{3}}{6} \cdot \left(2, \frac{1-\sqrt{3}}{2}, \frac{1+\sqrt{3}}{2}\right),$$

which is the permutation partner of the exact direct $(-s, c, b)$ shape. This coincidence lives at the factorised/projective level; it is not an exact identity of the centred additive preon components. Beyond this exact one-parameter II-1 family, a continuity-tracked scan of the local $\varepsilon < 0$ branches reveals one additional off-diagonal recurrence: on the I-$2/b_-$ branch, at

$$\varepsilon_\star \approx -0.01014665808,$$

the factorised reduced shape coincides with the exact I-1 shape, again up to permutation. Unlike the weaker drifts seen near the edge of a finite scan window, this is an interior recurrence and therefore appears to reflect a genuine algebraic relation rather than a boundary artefact. The same scan finds many exact sum and difference identities among $K(1/s, 1/c, 0)$ and $K(1/s, 1/c, \pm 1/t)$, but these largely reflect the fact that the centred inverse charges differ only by the small top entry. This inverse family is therefore structured, but not in a mysterious way: the algebra is real, yet much of it is inherited from the simple $0, \pm 1/t$ deformation of the same seed.

11.3 Pion-led composite patterns

The retained meson tuples of §11 can also be centred in the same way. If one interprets the common piece z_0 as a string-like background charge shared by the tuple, the pion-led direct tuples exhibit a fairly stable background scale together with a large negative pion residual:

Tuple	Type	preon masses [MeV]	Q	common piece	residuals
$(0, \pi^0, D^\pm)$	seed	$(334.4; 334.4, 44.46, 622.7)$	1.5012	$z_0 = 18.286$	$(-18.286, -6.668, 24.954)$
$(0, \pi^\pm, D_s)$	seed	$(350.7; 350.7, 47.79, 657.4)$	1.4973	$z_0 = 18.727$	$(-18.727, -6.913, 25.639)$
$(-\pi^0, D_s, \eta_c)$	direct	$(848.2; 1659.9, 232.3, 650.3)$	1.5006	$z_0 = 29.124$	$(-40.742, 15.242, 25.501)$
$(-\pi^\pm, D_s, B^\pm)$	direct	$(1229.9; 2198.2, 86.41, 1412.9)$	1.4984	$z_0 = 35.070$	$(-46.884, 9.296, 37.589)$
$(-\pi^\pm, D_s, B_s)$	direct	$(1244.0; 2216.9, 82.73, 1443.1)$	1.4978	$z_0 = 35.270$	$(-47.084, 9.096, 37.989)$
$(-\pi^0, D_s, B_c)$	direct	$(1392.9; 2395.1, 49.61, 1755.3)$	1.4962	$z_0 = 37.322$	$(-48.940, 7.044, 41.896)$
$(\pi^\pm, \pi^0, \eta_b)$	direct	$(1610.1; 801.6, 812.7, 3228.5)$	1.4981	$z_0 = 40.126$	$(-28.312, -28.508, 56.820)$
$K(1/\pi^0, 1/\eta_c, 1/W)$	inverse	$(772.9; 398.3, 3205.5, 950.1)$	1.5013	$w_0 = 0.036$	$(0.050, -0.018, -0.032)$

Table 9: Centred charge decompositions for the zero-entry pion seeds and the retained pion-led composite tuples. The seed rows isolate the $(0, \pi, D)$-type background before the sign-flipped heavy continuation to $(-\pi, D, X)$. The mass column lists the preon masses of the common and residual pieces.

Two qualitative features stand out.

- The zero-entry $(0, \pi, D)$-type seeds form a lower common-piece band, $z_0 \sim 18$–19 in $\mathrm{MeV}^{1/2}$ units, already near Koide. The retained $(-\pi, D, X)$ continuations sit in a distinct higher band $z_0 \sim 29$–40. Hence the pion sector does *not* exhibit a single universal background charge: the common piece depends strongly on whether one is looking at a seed tuple or at its heavy continuation.

- The direct pion tuples share a common pattern: the pion supplies the dominant negative residual, the middle meson supplies a modest positive one, and the heaviest state supplies the large positive counterweight. In this sense the pion behaves like the exceptional light constituent around which the rest of the tuple organizes.

- The direct pion families and the retained inverse tuple do *not* point to a single universal background scale: the direct families have $z_0 \sim 30$–40 in $\mathrm{MeV}^{1/2}$ units, whereas the retained inverse tuple has $w_0 \sim 0.036$ in $\mathrm{MeV}^{-1/2}$ units.

Taken together, the scan points to three robust regularities rather than a large zoo of numerology: the factor-3 exact scaling along the quark chain, the split pion common-piece bands separating the $(0, \pi, D)$ seeds from the $(-\pi, D, X)$ continuations, and the coherence of the inverse down-type family near $R_w = 1$. By contrast, the direct/inverse near-reciprocal hits occur only sporadically, often with an additional overall minus sign, so the present data do not support a single universal electric-magnetic or seesaw duality between all direct and inverse preon charges.

Among the retained meson benchmarks of §11, no kaon appears. Moreover, the current benchmark set does not distinguish the neutral-kaon basis $(K^0, \bar{K}^0)$ from the mass basis (K_S, K_L). If kaons are to participate in the same preon-pattern game, a dedicated scan in those alternative kaon bases is still needed.