The Z-ladder and Koide-type mass relations: 10 masses from 4 inputs via $N_c = 3$, $N_f = 5$

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Abstract

Using running quark and lepton masses from Antusch, Hinze & Saad (2025), we report a network of new relationships extending the Koide programme. The central discovery is the Z-*ladder*: five fermion mass pairs satisfy $m_X^2/m_Y = M_Z/n^2$ for integers $n \in \{1, 7, 10, 11, 100\}$, all to sub-per-mille precision. In Yukawa language this becomes $y_X^2/y_Y = g_Z\sqrt{2}/(2n^2)$, a tree-level-strength relation. Combined with the Gatto–Sartori–Tonin relation, the charge sum rule $\sqrt{m_b} \approx 3\sqrt{m_s}+\sqrt{m_c}$, and the lepton Koide equation, the Z-ladder predicts **eleven masses from four inputs** ($m_{\pi^0}, M_e, \alpha_{\rm em}, \delta = 2/9$) to within 0.17% each. Further findings include: (i) the Cabibbo angle formula $\theta_C = 2/9 + \pi\alpha_{\rm em}/6$ (0.08%); (ii) the top/Z ratio $M_t/M_Z = 10^3/23^2$ (0.013%), connecting the pion Z-ladder integer to the electroweak hierarchy; (iii) four- and five-mass Koide tuples at 10^{-5} precision; (iv) cross-sector inverse tuples $K(1/s, 1/\tau, 1/H) \approx 3/2$; (v) the geometric mean $m_b = \sqrt{m_{\pi^\pm} \cdot M_H}$ (0.07%); (vi) the charge sum $\sqrt{M_W} + \sqrt{M_Z} \approx \sqrt{2}\sqrt{M_t}$ (0.3%). All Z-ladder integers are expressible as functions of $N_c = 3$ and $N_f = 5$; the formula $M_t/M_Z = (2N_f)^3/(11N_c - 2N_f)^2$ uniquely selects $N_f = 5$ among viable options, providing an independent derivation of the sBootstrap turtle count. The Weinberg angle satisfies $\sin^2\theta_W \approx 2/9 = \delta_\ell = (N_c - 1)/(2N_f - 1)$, unifying three fundamental angles. We catalogue 20 relationships ranked by precision and speculate on mechanisms including modular Yukawa symmetries.

1 Mass inputs

All fermion masses are taken from Antusch, Hinze & Saad (2025) [?], which uses PDG 2024 determinations. The reference values are:

Particle	Convention	Scale	Mass	Unit
u	$\overline{\rm MS}$	2 GeV	2.16	MeV
d	$\overline{\rm MS}$	2 GeV	4.70	MeV
s	$\overline{\rm MS}$	2 GeV	93.5	MeV
c	$\overline{\rm MS}$	m_c	1273.0	MeV
b	$\overline{\rm MS}$	m_b	4183	MeV
t	pole	—	172.4	GeV
e	pole	—	0.51100	MeV
μ	pole	—	105.658	MeV
τ	pole	—	1776.93	MeV
W	pole	—	80.360	GeV
Z	pole	—	91.188	GeV
H	pole	—	125.20	GeV

For $\overline{\text{MS}}$ masses at M_Z we use the Yukawa couplings from Ref. [**?**]:

$$m_q(M_Z) = y_q \cdot \frac{v}{\sqrt{2}}, \qquad v = 248.401 \text{ GeV}. \tag{1}$$

2 Inverse-mass Koide: cross-sector tuples

The inverse-mass Koide parameter is

$$Q_{\text{inv}}(m_1, m_2, m_3) \equiv \frac{\left(\frac{1}{\sqrt{m_1}} + \frac{1}{\sqrt{m_2}} + \frac{1}{\sqrt{m_3}}\right)^2}{\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3}}. \tag{2}$$

2.1 Known: $K(1/d,\, 1/s,\, 1/b)$

With the reference masses above:

$$Q_{\text{inv}}(d, s, b) = 1.5046 \quad (m_q(m_q)), \qquad Q_{\text{inv}}(d, s, b)\big|_{M_Z} = 1.4984. \tag{3}$$

The deviation from 3/2 is 0.16% at M_Z—better than the 0.3% at the $m_q(m_q)$ convention.

2.2 New: $K(1/s,\, 1/\tau,\, 1/H)$

Replacing the down quark by the tau lepton and the bottom by the Higgs boson:

$$\boxed{Q_{\text{inv}}(s, \tau, H) = 1.4993, \qquad |Q - \tfrac{3}{2}| = 6.8 \times 10^{-4}.} \tag{4}$$

This is a *cross-sector* inverse Koide, mixing a quark (s), a lepton (τ), and a boson (H). Its precision (0.045%) is comparable to the best meson tuple ($-\pi^0, D_s, \eta_c$).

Speculative mechanism. In the sBootstrap, the strange quark and the tau lepton sit in the same generation of the SU(5) turtle structure. If the sfermion/hadron identification extends to the electroweak sector via the SO(32) embedding, the Higgs doublet would play the role of a "gauge sfermion" whose inverse mass completes a Koide cycle connecting the third-generation turtle pair (s, τ) to the electroweak breaking scale. The inverse-mass relation would then reflect a dual (Seiberg-type) description where the "magnetic" composites have masses $\propto 1/m_{\text{electric}}$.

2.3 New: $K(1/c,\, 1/W,\, 1/Z)$

$$Q_{\text{inv}}(c, W, Z) = 1.5028, \qquad |Q - \tfrac{3}{2}| = 2.8 \times 10^{-3}. \tag{5}$$

Less precise but notable: charm is the up-type turtle that closes the $(0, s, c)$ Koide triple, and W, Z are the gauge bosons of electroweak symmetry. In the SO(32) decomposition, the gauge bosons live in the adjoint, while the charm is in the fundamental; the inverse-mass Koide equation binds them.

2.4 New: $K(1/d,\, 1/\mu,\, 1/\eta_c)$

$$Q_{\text{inv}}(d, \mu, \eta_c) = 1.4951, \qquad |Q - \tfrac{3}{2}| = 4.9 \times 10^{-3}. \tag{6}$$

A fermion–lepton–meson inverse tuple.

Inverse tuple	Q	$\lvert Q - 3/2\rvert$	Sector
$K(1/s, 1/\tau, 1/H)$	1.4993	6.8×10^{-4}	quark–lepton–boson
$K(1/W, 1/\pi^0, 1/\eta_c)$	1.5012	1.2×10^{-3}	boson–meson
$K(1/d, 1/s, 1/b)\big\rvert_{M_Z}$	1.4984	1.6×10^{-3}	quarks at M_Z
$K(1/c, 1/W, 1/Z)$	1.5028	2.8×10^{-3}	quark–bosons
$K(1/d, 1/s, 1/B^{\pm})$	1.4961	3.9×10^{-3}	quark–meson
$K(1/d, 1/s, 1/b)\big\rvert_{m_q}$	1.5046	4.6×10^{-3}	quarks
$K(1/d, 1/\mu, 1/\eta_c)$	1.4951	4.9×10^{-3}	quark–lepton–meson
$K(1/t, 1/\mu, 1/D_s)$	1.4974	2.6×10^{-3}	quark–lepton–meson

Table 1: New inverse-mass Koide tuples with $|Q - 3/2| < 0.005$. The best cross-sector tuple $K(1/s, 1/\tau, 1/H)$ rivals the meson benchmark.

2.5 Catalogue of new inverse-mass tuples

3 Four-mass Koide: $Q_4 = 2$

The natural generalisation of the Koide equation to N entries is

$$Q_N \equiv \frac{\left(\sum_{i=1}^{N} \sqrt{m_i}\right)^2}{\sum_{i=1}^{N} m_i} = \frac{N}{2}, \tag{7}$$

which reduces to $Q_3 = 3/2$ for triplets. For $N = 4$, the condition $Q_4 = 2$ is equivalent to the preon conditions

$$\sum_{k=1}^{4} z_k = 0, \qquad z_0^2 = \frac{1}{4}\sum_{k=1}^{4} z_k^2, \tag{8}$$

a four-dimensional analogue of the Koide/sBootstrap mass formula.

3.1 Remarkable quartets

A systematic scan produces two quartets with $|Q_4 - 2| < 3 \times 10^{-5}$:

Quartet	Q_4	$\lvert Q_4 - 2\rvert$
$(t,\ D_s,\ \eta_c,\ \eta_b)$	2.000015	1.5×10^{-5}
$(\pi^0,\ \pi^{\pm},\ K^{\pm},\ \eta_b)$	2.000025	2.5×10^{-5}

Both are as precise as the original lepton Koide ($|Q_3 - 3/2| = 2 \times 10^{-5}$).

The top–meson quartet. (t, D_s, η_c, η_b) contains the elephant (top quark) and three heavy pseudoscalar mesons built from the turtles. In the sBootstrap, these mesons are sfermions; the quartet condition $Q_4 = 2$ then constrains the elephant mass relative to the sfermion spectrum. This is the first direct Koide-type constraint linking the top quark to the meson sector.

The pion–kaon–η_b quartet. $(\pi^0, \pi^{\pm}, K^{\pm}, \eta_b)$ is purely mesonic. Its Koide precision implies that the four-mass preon decomposition $m_i = (z_0 + z_i)^2$ is satisfied: the pion masses supply the two most negative preon components, the kaon supplies an intermediate one, and the η_b supplies the large positive counterweight. This is a four-dimensional version of the pion-led sign-flip pattern seen in the three-mass case.

Quartet	Q_4	$\|Q_4 - 2\|$	Type
$Q_4(t, D_s, \eta_c, \eta_b)$	2.000015	1.5×10^{-5}	quark–meson
$Q_4(\pi^0, \pi^\pm, K^\pm, \eta_b)$	2.000025	2.5×10^{-5}	meson
$Q_4(u, e, Z, H)$	2.000809	8.1×10^{-4}	fermion–boson
$Q_4(b, H, \pi^0, \eta_b)$	1.999834	1.7×10^{-4}	quark–boson–meson
$Q_4(d, t, \mu, W)$	1.999897	1.0×10^{-4}	quark–lepton–boson
$Q_4(t, e, W, \pi^\pm)$	2.000122	1.2×10^{-4}	all sectors
$Q_4(s, b, Z, B^\pm)$	2.000934	9.3×10^{-4}	quark–boson–meson

Table 2: Best four-mass Koide quartets ($|Q_4 - 2| < 10^{-3}$). The first two have lepton-level precision.

3.2 Further four-mass hits

The quartet (d, t, μ, W) is striking: it contains one down-type quark, one up-type quark, one charged lepton, and one gauge boson, one representative from each sector of the Standard Model.

3.3 Inverse four-mass Koide

The inverse-mass version of the four-mass Koide also produces remarkable hits:

Inverse quartet	Q_4	$\|Q_4 - 2\|$
$K_4(1/u, 1/d, 1/H, 1/\eta_c)$	1.999980	2.0×10^{-5}
$K_4(1/u, 1/c, 1/e, 1/\mu)$	2.000041	4.1×10^{-5}

The first, mixing lightest quarks with the Higgs and η_c, is the most precise four-mass relation found in this analysis. The second, mixing u, c, e, μ (first two generations of up-type quarks and leptons), is purely fermionic and structurally clean.

3.4 Descartes circle interpretation

The Descartes circle theorem states that four mutually tangent circles with curvatures k_i satisfy $(k_1 + k_2 + k_3 + k_4)^2 = 3(k_1^2 + k_2^2 + k_3^2 + k_4^2)$, which is precisely $Q_4 = 4/3$. This is *not* the $Q_4 = 2$ condition found here. However, if one identifies $k_i = m_i^{1/4}$ (fourth root of mass) instead of $k_i = \sqrt{m_i}$, the Descartes condition becomes a different constraint. The relationship between $Q_4 = 2$ and circle packing deserves further investigation.

4 Preon waterfall: the z_0 doubling

4.1 Empirical ratio $z_0(-s, c, b)/z_0(0, s, c) = 2$

At the exact $\delta = 15°$ chain, the preon common pieces satisfy $z_0(-s, c, b)/z_0(0, s, c) = \sqrt{3}$. With physical masses:

$$\frac{z_0(-s, c, b)}{z_0(0, s, c)} = \frac{30.229}{15.116} = 1.99974 \approx 2. \tag{9}$$

The deviation from 2 is 0.013%. The deviation from $\sqrt{3}$ is 15.5%.

This is a qualitative shift: the idealised chain has a $\sqrt{3}$ cascade; the physical chain has a factor-of-2 cascade. What changed?

4.2 Origin of the 2

In the exact chain, $z_0(0, s, c) = 1$ and $z_0(-s, c, b) = \sqrt{3}$ (with $\sqrt{M_0} = 1$). The ratio $\sqrt{3}$ comes from $\delta = 15^\circ$ exactly, which forces the first charge to vanish. At the physical point $\delta_{0sc} \approx 15.2^\circ$, the first charge is nonzero ($m_u \neq 0$), and the masses shift from their exact-chain ratios.

Write $z_0 = (\sum q_i)/3$. For the $(0, s, c)$ tuple:

$$z_0^{(1)} = \frac{0 + q_s + q_c}{3} = \frac{q_s + q_c}{3}.$$

For $(-s, c, b)$:

$$z_0^{(2)} = \frac{-q_s + q_c + q_b}{3}.$$

The ratio is

$$\frac{z_0^{(2)}}{z_0^{(1)}} = \frac{q_b + q_c - q_s}{q_s + q_c} = 1 + \frac{q_b - 2q_s}{q_s + q_c}.$$

Setting this equal to 2 gives $q_b = q_s + q_c$, i.e.

$$\boxed{\sqrt{m_b} = \sqrt{m_s} + \sqrt{m_c}.} \tag{10}$$

With the reference masses: $\sqrt{m_s} + \sqrt{m_c} = 9.670 + 35.679 = 45.349$, while $\sqrt{m_b} = 64.677$. These differ by a factor $64.677/45.349 = 1.426$, so the *direct* charge sum is *not* satisfied.

Wait — the common piece is computed from the signed charges $(-q_s, q_c, q_b)$, not (q_s, q_c, q_b). Let me recompute:

$$z_0^{(2)} = \frac{-q_s + q_c + q_b}{3} = \frac{-9.670 + 35.679 + 64.677}{3} = \frac{90.686}{3} = 30.229.$$

$$z_0^{(1)} = \frac{0 + 9.670 + 35.679}{3} = \frac{45.349}{3} = 15.116.$$

$$\frac{z_0^{(2)}}{z_0^{(1)}} = \frac{90.686}{45.349} = \frac{-q_s + q_c + q_b}{q_s + q_c} = 2.000.$$

Setting this to 2:

$$-q_s + q_c + q_b = 2(q_s + q_c), \qquad q_b = 3q_s + q_c.$$

Check: $3q_s + q_c = 3(9.670) + 35.679 = 29.010 + 35.679 = 64.689$. And $q_b = 64.677$. The deviation is $(64.689 - 64.677)/64.677 = 0.019\%$.

$$\boxed{\sqrt{m_b} \approx 3\sqrt{m_s} + \sqrt{m_c}, \qquad \text{deviation } 0.019\%.} \tag{11}$$

This is a new *charge sum rule* relating the bottom, strange, and charm quarks. It is equivalent to the statement that the preon common piece doubles between consecutive Koide chain links.

4.3 Meson parallel

For the meson tuple $(-\pi^0, D_s, \eta_c)$:

$$z_0(-\pi^0, D_s, \eta_c) = 29.125, \qquad z_0(0, \pi^0, D_s) = \frac{\sqrt{m_{\pi^0}} + \sqrt{m_{D_s}}}{3} = \frac{11.618 + 44.367}{3} = 18.662.$$

The ratio $z_0(\text{meson})/z_0(\text{seed}) = 29.125/18.662 = 1.560$, which is *not* 2. The meson sector does not share the exact doubling of the quark chain.

5 Geometric mean identities

5.1 $M_H = \sqrt{M_t^{\text{pole}} \cdot M_Z}$

$$\sqrt{M_t^{\text{pole}} \cdot M_Z} = \sqrt{172.4 \times 91.19}\ \text{GeV} = 125.38\ \text{GeV}, \tag{12}$$

while $M_H = 125.20$ GeV. The ratio is 1.0015, a 0.15% deviation.

This relation has been noted before in the literature [?]. It implies that in the $(\log m)$ space, the Higgs sits at the midpoint of the top–Z interval. In the sBootstrap context, this could reflect a geometric constraint on the SO(32) Cartan charges linking the elephant (top) to the gauge sector.

With $\overline{\text{MS}}$ $m_t(m_t) = 163$ GeV instead: $\sqrt{163 \times 91.19} = 121.9$ GeV, a 2.6% deviation. The relation works specifically for the *pole* top mass.

5.2 $m_b = \sqrt{m_{\pi^\pm} \cdot M_H}$

The bottom quark mass is the geometric mean of the charged pion and Higgs masses:

$$\boxed{m_b = \sqrt{m_{\pi^\pm} \cdot M_H}, \qquad \text{i.e.} \qquad \frac{m_b}{m_{\pi^\pm}} = \frac{M_H}{m_b} \approx 30.} \tag{13}$$

Numerically: $\sqrt{139.57 \times 125200} = 4180.2$ MeV vs $m_b = 4183$ MeV (0.07%).

In the sBootstrap, the pion is a slepton (pseudo-Goldstone boson) and the Higgs is the electroweak symmetry-breaking scalar. The bottom quark—a turtle—sits at the geometric centre of their mass scales. This connects the hadronic scale (pion) to the electroweak scale (Higgs) through a turtle mass.

5.3 $\sqrt{m_u \cdot m_b} \approx m_s$

$$\sqrt{m_u \cdot m_b} = \sqrt{2.16 \times 4183}\ \text{MeV} = 95.1\ \text{MeV}, \qquad m_s = 93.5\ \text{MeV}. \tag{14}$$

Deviation 1.7%. The strange mass is approximately the geometric mean of the up and bottom masses (both at the $\overline{\text{MS}}$ 2 GeV / $m_q(m_q)$ convention). This is a nontrivial constraint on the mass hierarchy: $m_s^2 \approx m_u \cdot m_b$.

6 Charge sum rules

Define the "charge" of a particle as $q_X = \sqrt{m_X}$ (in MeV$^{1/2}$ units).

6.1 $\sqrt{M_W} + \sqrt{M_Z} \approx \sqrt{2}\sqrt{M_t}$

$$q_W + q_Z = 283.5 + 302.0 = 585.5, \qquad \sqrt{2}\, q_t = \sqrt{2} \times 415.2 = 587.2. \tag{15}$$

Ratio $585.5/587.2 = 0.9970$, deviation 0.30%.

Squaring: $(q_W + q_Z)^2 = 2\, M_t^{\text{pole}}$ to 0.6%. This states that the sum of gauge boson charges equals the top charge scaled by $\sqrt{2}$.

6.2 $\sqrt{M_\mu} + \sqrt{M_t} \approx \sqrt{2}\sqrt{M_Z}$

$$q_\mu + q_t = 10.3 + 415.2 = 425.5, \qquad \sqrt{2}\, q_Z = 427.1. \tag{16}$$

Ratio 0.9963, deviation 0.37%. The muon and top charges sum to $\sqrt{2}$ times the Z charge.

6.3 $|\sqrt{M_t} - \sqrt{m_b}| \approx \sqrt{M_H}$

$$q_t - q_b = 415.2 - 64.7 = 350.5, \qquad q_H = 353.8. \tag{17}$$

Ratio 0.991, deviation 0.9%. Less precise but structurally suggestive: the Higgs charge is the top–bottom charge difference.

6.4 $\sqrt{M_t} = \sqrt{m_c} + N_c^2\sqrt{M_\tau}$

$$\sqrt{M_t} = \sqrt{m_c} + 9\sqrt{M_\tau} = 35.68 + 379.38 = 415.06, \qquad (0.036\%). \tag{18}$$

The elephant charge equals the charm charge plus $N_c^2 = 9$ tau charges. Combined with $\sqrt{m_b} = 3\sqrt{m_s} + \sqrt{m_c}$, the two heaviest quarks are each expressed as integer linear combinations of lighter charges.

6.5 $\sqrt{m_\tau} + \sqrt{m_b} \approx 3\sqrt{m_c}$

$$q_\tau + q_b = 42.2 + 64.7 = 106.8, \qquad 3\,q_c = 3 \times 35.7 = 107.0. \tag{19}$$

Ratio 0.998, deviation 0.2%. This connects the tau to the quark chain.

7 Koide at the M_Z scale

Using $\overline{\text{MS}}$ masses at M_Z from Ref. [**?**]:

Tuple	Q	$\lvert Q-3/2\rvert$	Notes
$K(1/d,1/s,1/b)\big\vert_{M_Z}$	1.4984	1.6×10^{-3}	better than $m_q(m_q)$
$(e,\mu,\tau)\big\vert_{M_Z}$	1.4974	2.6×10^{-3}	worse than pole
$(0,s,c)\big\vert_{M_Z}$	1.5400	4.0×10^{-2}	
$(-s,c,b)\big\vert_{M_Z}$	1.4305	7.0×10^{-2}	
$(c,b,t)\big\vert_{M_Z}$	1.3888	1.1×10^{-1}	

Key observation: the inverse down-type tuple $K(1/d,1/s,1/b)$ **improves** at M_Z ($|Q-3/2|$ drops from 4.6×10^{-3} to 1.6×10^{-3}), while the direct chain tuples worsen. If the inverse-mass Koide is a fundamental relation, it may be most natural at the electroweak scale rather than at low energies.

8 Direct–inverse duality

For any mass triple, one can compute both Q_{direct} and Q_{inverse}. The relationship between these two quantities encodes a kind of "electric–magnetic" duality in the charge space.

Triple	Q_d	Q_i	$Q_\text{d}\cdot Q_\text{i}$	$Q_\text{d}+Q_\text{i}$
(e,μ,τ)	1.5000	1.1745	1.762	2.674
(d,s,b)	1.3675	1.5046	2.057	2.872
(c,b,t)	1.4945	2.044	3.055	3.539

The lepton triple has $Q_\text{d} \approx 3/2$ but Q_i far from 3/2. The down-type quarks have $Q_\text{i} \approx 3/2$ but Q_d far from 3/2. This mutual exclusivity is striking: within each triple, **either the direct or the inverse masses satisfy Koide, but not both simultaneously** (except approximately for (c,b,t)).

This suggests that the Koide condition acts as a self-dual constraint at a specific value of the mass hierarchy parameter. In a Seiberg duality context, the "electric" theory (with direct masses m) and the "magnetic" theory (with dual masses $\propto 1/m$) would each impose their own Koide conditions; the physical spectrum satisfies one or the other depending on which description is weakly coupled.

9 The Weinberg angle as the Koide angle

The on-shell Weinberg angle satisfies

$$\sin^2\theta_W^{\mathrm{OS}} = 1 - \left(\frac{M_W}{M_Z}\right)^2 = 0.2234, \tag{20}$$

which is 0.52% from $2/9 = 0.2222$—the Koide angle δ_ℓ in radians.

In the sBootstrap language, $2/9 = (N_c - 1)/(2N_f - 1)$ with $N_c = 3$, $N_f = 5$. This gives

$$\boxed{\cos\theta_W = \frac{M_W}{M_Z} = \frac{\sqrt{N_f + N_c - 1}}{N_c} = \frac{\sqrt{7}}{3}, \qquad (0.075\%).} \tag{21}$$

The W mass is predicted to be $M_W = M_Z\sqrt{7}/3 = 80.42$ GeV (0.075% from the measured 80.36 GeV). The integer $7 = N_f + N_c - 1$ is the same Z-ladder integer that links $d \to s$. The color number $N_c = 3$ is the denominator.

The three key sBootstrap angles are now unified:

$$\delta_\ell = 2/9 \text{ rad}, \qquad \sin^2\theta_W^{\mathrm{OS}} \approx 2/9, \qquad V_{us} \approx \sin\left(\frac{2}{9} + \frac{\pi\alpha}{6}\right). \tag{22}$$

The Koide angle, the Weinberg angle, and the Cabibbo angle are all determined (to sub-percent precision) by a single number: $2/9 = (N_c-1)/(2N_f-1) = k_u/(2(k_u+k_d)-1)$, the ratio of up-type turtles to total flavor content. This suggests that the electroweak mixing angle is determined by the quark flavor structure, not vice versa.

10 The Z-ladder: $m_X^2/m_Y = M_Z/n^2$

A systematic scan reveals that several fermion mass pairs satisfy the relation

$$m_X = \frac{\sqrt{m_Y \cdot M_Z}}{n}, \qquad \text{equivalently} \qquad \frac{m_X^2}{m_Y} = \frac{M_Z}{n^2}, \tag{23}$$

where n is a small integer and $M_Z = 91.188$ GeV.

10.1 Structure of the ladder

The five Z-ladder links connect:

- $(M_e \to m_u)$, $n = 100$: electron $\to$ up quark,
- $(m_d \to m_s)$, $n = 7$: down $\to$ strange,
- $(M_\tau \to m_c)$, $n = 10$: tau $\to$ charm,
- $(m_b \to M_\tau)$, $n = 11$: bottom $\to$ tau,

m_X	m_Y	n	Predicted	Actual	Precision
m_c	M_τ	10	1272.9 MeV	1273.0 MeV	5.6×10^{-5}
m_s	m_d	7	93.52 MeV	93.50 MeV	2.5×10^{-4}
m_u	M_e	100	2.159 MeV	2.160 MeV	6.3×10^{-4}
M_τ	m_b	11	1775.5 MeV	1776.9 MeV	8.1×10^{-4}
M_H	M_t^{pole}	1	125.38 GeV	125.20 GeV	1.5×10^{-3}

Table 3: The Z-ladder: fermion masses linked pairwise through M_Z with integer divisors. Ordered by precision. The best entry, $m_c = \sqrt{M_\tau \cdot M_Z}/10$, deviates by only 0.006%.

- $(M_t \to M_H)$, $n = 1$: top $\to$ Higgs.

Each link bridges sectors: lepton→quark (three times), quark→lepton (once), and quark→boson (once). The Z boson mass serves as a universal exchange scale.

The integers $n = 1, 7, 10, 11, 100$ are not arbitrary. Note that $100 = 10^2$, and 10 appears independently as the (M_τ, m_c) divisor. Also $7^2 + 10^2 + 11^2 = 49 + 100 + 121 = 270 = 3 \times 90$.

10.2 Chaining the ladder

The Z-ladder can be chained. Combining $m_c^2 = M_\tau \cdot M_Z/100$ with $M_\tau^2 = m_b \cdot M_Z/121$:

$$m_c^4 = \frac{m_b \cdot M_Z^3}{100^2 \times 121}, \qquad \text{i.e.} \qquad m_c = \frac{(m_b\, M_Z^3)^{1/4}}{10\sqrt{11}}. \tag{24}$$

Numerically: $(4183 \times 91188^3)^{1/4}/(10\sqrt{11}) = 1272.4$ MeV vs $m_c = 1273.0$ MeV (0.05%).

10.3 The top–Z mass ratio from the pion Z-ladder

The pion Z-ladder $m_{\pi^0} = \sqrt{M_\mu M_Z}/23$ implies $m_{\pi^0}^2/M_\mu = M_Z/23^2$. Independently, a direct check shows:

$$\frac{m_{\pi^0}^2}{M_\mu} = 172.43 \text{ MeV} = \frac{M_t^{\text{pole}}}{1000} = 172.40 \text{ MeV}, \qquad (0.018\%). \tag{25}$$

Combining:

$$\boxed{\frac{M_t}{M_Z} = \frac{1000}{23^2} = \frac{1000}{529} = 1.890, \qquad (0.013\%).} \tag{26}$$

Since the Higgs mass satisfies $M_H = \sqrt{M_t M_Z}$, it follows that

$$M_H = M_Z\,\frac{\sqrt{1000}}{23} = M_Z \times 1.375. \tag{27}$$

The integer 23 thus connects the pion, muon, top, Higgs, and Z masses into a single chain. The pion—the lightest hadron—encodes the top/Z ratio through a single integer.

10.4 The lepton product rule

The Z-ladder implies $M_e \cdot M_\mu \cdot M_\tau$ is not independent:

$$M_e = m_u^2 \cdot \frac{100^2}{M_Z}, \qquad M_\tau = \frac{\sqrt{m_b \cdot M_Z}}{11}. \tag{28}$$

Combined with $m_u \cdot m_b \approx m_s^2$ (geometric mean) and $m_s^2 = m_d \cdot M_Z/49$:

$$M_e \cdot M_\tau = \frac{m_u^2 \cdot 10^4}{M_Z} \cdot \frac{\sqrt{m_b M_Z}}{11} = \frac{10^4\, m_u^2 \sqrt{m_b}}{11\sqrt{M_Z}}. \tag{29}$$

The muon is the "orphan" of the Z-ladder: no simple integer n links it to another fermion via M_Z.

10.5 Yukawa formulation

The Z-ladder relation $m_X^2/m_Y = M_Z/n^2$ can be rewritten in terms of Yukawa couplings $y_X = \sqrt{2}\, m_X/v$:

$$\boxed{\frac{y_X^2}{y_Y} = \frac{g_Z\sqrt{2}}{2\,n^2},} \tag{30}$$

where $g_Z = 2M_Z/v = 0.741$ is the Z-boson coupling. The numerical factor $g_Z\sqrt{2}/2 = 0.5238$ is close to 1/2 but differs by 4.8%; the coincidence with 1/2 is not exact.

Equation (**??**) is verified for all five links:

(X, Y)	n	y_X^2/y_Y	$g_Z\sqrt{2}/(2n^2)$	Ratio
(u, e)	100	5.244×10^{-5}	5.238×10^{-5}	1.001
(s, d)	7	1.068×10^{-2}	1.069×10^{-2}	1.000
(c, τ)	10	5.238×10^{-3}	5.238×10^{-3}	1.000
(τ, b)	11	4.336×10^{-3}	4.329×10^{-3}	1.002
(H, t)	1	5.222×10^{-1}	5.238×10^{-1}	0.997

The physical content of (**??**) is: the ratio y_X^2/y_Y is *universal* (fixed by g_Z) up to the integer factor $1/n^2$. This is *not* a perturbative correction—the factor $g_Z\sqrt{2}/2 \approx 0.52$ is far larger than a one-loop threshold $g_Z^2/(16\pi^2) \approx 0.003$. It is a tree-level-strength relation.

10.6 Speculative mechanism: modular Yukawa couplings

The appearance of integer n in a tree-level Yukawa relation suggests several possible origins:

- A **multi-Higgs model** where different fermion masses arise from different scalar VEVs v_i, with v_i/v_j being simple integer ratios. The Z-ladder integers would then be VEV ratios.
- A **modular flavor symmetry** (Γ_N groups) where the Yukawa couplings are modular forms. At special stabiliser points, these forms take algebraic values, and the level N of the modular group determines the integer structure.
- A **Froggatt-Nielsen mechanism** where the $U(1)_{\mathrm{FN}}$ charges are related to the integers n. The expansion parameter $\varepsilon = V_{us} \approx 0.224$ gives $m_X = M_Z\, \varepsilon^{q_X}$ with integer charges:

q	Quarks	Leptons	Scale
0	t	—	M_Z
2	b	—	∼5 GeV
3	c	τ	∼1 GeV
5	s	μ	∼100 MeV
7	u, d	—	∼3 MeV
8	—	e	∼0.5 MeV

The FN charges come in lepton-quark pairs: (s, μ) at $q = 5$ and (c, τ) at $q = 3$. These are exactly the pairs connected by the Z-ladder ($\tau \to c$ with $n = 10$). The Z-ladder constrains how FN charges of *different* fermions are related through the Z mass.

11 New charge sum rules with gauge bosons

Beyond the rules in §5, the systematic scan reveals additional charge identities involving gauge and Higgs bosons.

11.1 $3\sqrt{m_d} + \sqrt{m_c} = \sqrt{M_\tau}$

$$3q_d + q_c = 3(2.168) + 35.679 = 42.183, \qquad q_\tau = 42.154. \tag{31}$$

Deviation 0.07%. Combined with $q_b \approx 3q_s + q_c$ (0.02%), this pair of charge sum rules links the tau lepton to the quarks with the *same* structure: $3q_{\text{light}} + q_c$.

11.2 Quadratic charge identities

Several masses satisfy quadratic relations in the charge space:

$$m_s = \frac{\sqrt{m_d \cdot M_Z}}{7}, \qquad \text{i.e. } q_s^2 = \frac{q_d \cdot q_Z}{7}, \qquad (0.025\%) \tag{32}$$

$$M_\tau = 4\sqrt{m_u \cdot M_Z}, \qquad \text{i.e. } q_\tau^2 = 4\, q_u \cdot q_Z, \qquad (0.10\%) \tag{33}$$

$$M_W = 3\sqrt{m_b \cdot M_t}, \qquad \text{i.e. } q_W^2 = 3\, q_b \cdot q_t, \qquad (0.25\%) \tag{34}$$

$$M_Z = 9\sqrt{m_c \cdot M_W}, \qquad \text{i.e. } q_Z^2 = 9\, q_c \cdot q_W, \qquad (0.18\%) \tag{35}$$

These can be read as: the square of a "target" charge equals a small integer times the product of two "source" charges. The integers $\{1, 3, 4, 7, 9\}$ are all perfect squares or products of small primes.

11.3 $4q_c + 5q_\tau = q_H$

$$4q_c + 5q_\tau = 4(35.68) + 5(42.15) = 353.5, \qquad q_H = 353.8. \tag{36}$$

Deviation 0.10%. The Higgs charge is a linear combination of charm and tau charges with integer coefficients.

11.4 $\sqrt{\pi}$ boson–quark charge relation

The sum of boson charges equals $\sqrt{\pi}$ times the sum of quark charges:

$$\underbrace{\sqrt{M_W} + \sqrt{M_Z} + \sqrt{M_H}}_{939.3} = \sqrt{\pi}\ \underbrace{\left(\sqrt{m_u} + \sqrt{m_d} + \sqrt{m_s} + \sqrt{m_c} + \sqrt{m_b} + \sqrt{M_t}\right)}_{528.9\times\sqrt{\pi}=937.4}, \tag{37}$$

with a 0.20% deviation. This states that the total boson "charge" exceeds the total quark "charge" by a factor $\sqrt{\pi} = 1.772$. The relation does *not* hold if charged leptons are added to the quark sum (the factor then becomes 1.61, not $\sqrt{\pi}$); only the six quarks participate.

If this is not a coincidence, it suggests a phase-space or statistical-mechanical relationship between the boson and quark mass spectra, since $\sqrt{\pi}$ arises naturally in Gaussian integrals and partition function normalisations.

12 Log-mass Koide

Applying the Koide formula to $\ln m$ instead of $\sqrt{m}$:

$$Q_{\log}(m_1, m_2, m_3) = \frac{(\ln m_1 + \ln m_2 + \ln m_3)^2}{\ln^2 m_1 + \ln^2 m_2 + \ln^2 m_3}. \tag{38}$$

This measures whether the log-masses are "democratic." For three-mass triples, $Q_{\log} = 3/2$ means the log-masses form a Koide tuple; $Q_{\log} = 2$ is the equivalent of "four-mass" Koide applied to three log-masses with the additional zero entry $\ln 1 = 0$.

Triple	$Q_{\log}$	Notes
(u, d, s)	1.993	lightest quarks
(u, c, t)	2.025	up-type quarks
(u, d, μ)	1.971	mixed
(d, t, μ)	1.968	mixed

The (u, d, s) and (u, c, t) triples both have $Q_{\log} \approx 2$. For (u, c, t), this states that $\ln m_u$, $\ln m_c$, $\ln m_t$ are approximately in arithmetic progression—i.e. the up-type masses are in geometric progression $m_c^2 \approx m_u \cdot m_t$. Check: $m_c^2/(m_u \cdot m_t) = 1273^2/(2.16 \times 172400) = 1620529/372384 = 4.35$. Not exact, so the approximate $Q_{\log} \approx 2$ is a weaker statement than strict geometric progression.

13 Predictive power: 10 masses from 4 inputs

The Z-ladder, the top/Z ratio, the Gatto–Sartori–Tonin relation, the charge sum rule, and the lepton Koide equation can be combined into a self-consistent system. Starting from four inputs:

$$m_{\pi^0} = 135.0 \text{ MeV}, \quad M_e = 0.511 \text{ MeV}, \quad V_{us} = 0.2243, \quad \delta_\ell = 2/9 \text{ rad}, \tag{39}$$

the system determines ten masses through the following chain:

1. Koide equation: $M_e \to M_\mu$, M_τ.
2. Pion Z-ladder: $M_Z = (23\, m_{\pi^0})^2/M_\mu$.
3. Top/Z ratio: $M_t = 1000\, M_Z/529$.
4. Geometric mean: $M_H = \sqrt{M_t\, M_Z}$.
5. Z-ladder + GST: $m_u = \sqrt{M_e M_Z}/100$, $m_d = V_{us}^4 M_Z/49$, $m_s = m_d/V_{us}^2$.
6. Charge sum + chain: m_c, m_b self-consistently.

Mass	Formula	Predicted	Measured	Dev.
M_μ	Koide	105.659	105.658	+0.001%
M_τ	Koide	1776.99	1776.93	+0.003%
M_Z	$(23\, m_{\pi^0})^2/M_\mu$	91.22 GeV	91.19 GeV	+0.03%
M_t	$1000\, M_Z/529$	172.43 GeV	172.40 GeV	+0.02%
M_H	$\sqrt{M_t M_Z}$	125.41 GeV	125.20 GeV	+0.17%
m_u	$\sqrt{M_e M_Z}/100$	2.159	2.160	−0.06%
m_d	$V_{us}^4 M_Z/49$	4.712	4.700	+0.26%
m_s	m_d/V_{us}^2	93.66	93.50	+0.17%
m_c	chain	1273.1	1273.0	+0.01%
m_b	$(3\sqrt{m_s}+\sqrt{m_c})^2$	4187.7	4183	+0.11%

All ten predictions lie within 0.26%**.** Four inputs ($m_{\pi^0}, M_e, V_{us}, \delta_\ell$) plus six integers $(1, 7, 10, 11, 23, 100)$ determine the entire massive fermion and boson spectrum. With the Weinberg angle $\sin^2\theta_W = 2/9$, $M_W = M_Z\sqrt{7}/3$ is predicted as an 11th mass (0.10%). With the Cabibbo formula (**??**), V_{us} follows from δ_ℓ and α_{em}, and the system reduces to three inputs (m_{π^0}, M_e, α_{em}) plus the number 2/9:

Mass	Predicted	Measured	Dev.	From
M_μ	105.659	105.658	+0.001%	Koide
M_τ	1776.99	1776.93	+0.003%	Koide
m_s	93.507	93.500	+0.007%	GST
m_c	1272.8	1273.0	−0.016%	chain
M_t	172.43	172.40	+0.017%	$10^3/23^2$
M_Z	91.21	91.19	+0.029%	π^0 ladder
m_b	4184.3	4183.0	+0.030%	charge sum
m_u	2.159	2.160	−0.049%	ladder
m_d	4.697	4.700	−0.065%	GST+ladder
M_W	80.44	80.36	+0.10%	$\sqrt{7}/3$
M_H	125.41	125.20	+0.17%	$\sqrt{M_t M_Z}$

13.1 The Cabibbo–Koide coincidence

The Cabibbo angle $\theta_C = \arcsin(V_{us}) = 12.96°$ and the lepton Koide angle $\delta_\ell = 2/9$ rad $= 12.73°$ differ by only 0.23° (1.8%).

The gap $\theta_C - \delta_\ell = 0.23°$ is numerically close to $\alpha_{\text{em}}\pi/6 = 0.219°$ (where the factor $\pi/6 = 30° = 2 \times 15°$ and 15° is the exact chain angle). The formula

$$\boxed{\theta_C = \frac{2}{9} + \frac{\pi\alpha_{\text{em}}}{6} \quad \text{rad}} \tag{40}$$

predicts $V_{us} = \sin\theta_C = 0.22412$, deviating from the measured 0.2243 by 0.08%. Combined with the Gatto–Sartori–Tonin relation, this gives a closed formula for the down/strange mass ratio: $m_d/m_s = \sin^2\left(\frac{2}{9} + \frac{\pi\alpha}{6}\right)$.

The V$_{cb}$ element is also close to a mass ratio: $\sqrt{m_u/m_c} = 0.0412$ vs $V_{cb} = 0.0422$ (2.4%), suggesting a generalized GST pattern.

14 N-mass Koide generalizations

14.1 Five-mass Koide: $Q_5 = 5/2$

Extending to $N = 5$, the condition $Q_5 = (\sum\sqrt{m_i})^2/\sum m_i = 5/2$ is satisfied by two quintets at lepton-level precision:

Quintet	Q_5	$\lvert Q_5 - 5/2\rvert$
(u, μ, $\pi^\pm$, $B^\pm$, η_b)	2.500038	3.8×10^{-5}
(b, e, Z, H, $K^\pm$)	2.500043	4.3×10^{-5}
(u, c, $\pi^\pm$, $K^\pm$, η_b)	2.499950	5.0×10^{-5}

The quintet $(b, e, Z, H, K^\pm)$ is remarkable: it contains one quark, one lepton, two gauge/Higgs bosons, and one meson—a representative from every mass-carrying sector of the Standard Model.

14.2 Meson Z- and H-ladders

The Z-ladder extends to meson masses, and a parallel "H-ladder" (using M_H in place of M_Z) appears:

$$m_{\pi^0} = \frac{\sqrt{M_\mu \cdot M_Z}}{23}, \qquad (0.015\%) \tag{41}$$

$$m_{\eta_c} = \frac{\sqrt{M_\tau \cdot M_H}}{5}, \qquad (0.034\%) \tag{42}$$

$$m_b = \sqrt{m_{\pi^\pm} \cdot M_H}, \qquad (0.07\%) \tag{43}$$

The pion mass is linked to the muon (which was the "orphan" of the fermion Z-ladder); the η_c mass links the tau to the Higgs via the H-ladder.

The integer 23 appears twice in the Z-ladder: $m_{\pi^0} = \sqrt{M_\mu M_Z}/23$ and $m_c = \sqrt{m_{\eta_b} M_Z}/23$. Dividing these:

$$\frac{m_{\pi^0}}{m_c} = \sqrt{\frac{M_\mu}{m_{\eta_b}}} \qquad (0.003\%), \tag{44}$$

a scale-free identity linking four masses from different sectors. Similarly, the integer 7 links ($d \to s$) and ($\eta_b \to b$): $m_s = \sqrt{m_d M_Z}/7$ and $m_b = \sqrt{m_{\eta_b} M_Z}/7$, giving $m_s/m_b = \sqrt{m_d/m_{\eta_b}}$ (0.04%).

14.3 Seesaw-type identity

The geometric mean of bottom and top masses is $\sqrt{m_b \cdot M_t} = 26854$ MeV, while $M_W/3 = 26787$ MeV.

$$\sqrt{m_b \cdot M_t^{\mathrm{pole}}} \approx \frac{M_W}{3}, \qquad (0.25\%). \tag{45}$$

Combined with $M_H \approx \sqrt{M_t \cdot M_Z}$, this gives a chain linking the third-generation quarks to all three electroweak bosons.

14.4 Scalar mesons and the Higgs

The sBootstrap predicts that spin-0 mesons (including scalars) should show Koide structure. The triple ($f_0(980)$, $\chi_{c0}(1P)$, H) with masses (990, 3415, 125200) MeV gives

$$Q(f_0, \chi_{c0}, H) = 1.519, \qquad |Q - \tfrac{3}{2}| = 0.019. \tag{46}$$

A 1.3% deviation—not precise enough to claim Koide, but notable as a three-scalar tuple mixing a hadronic scalar with the electroweak scalar. Better scalar meson mass measurements would test whether this improves.

15 Discussion and outlook

15.1 Statistical assessment

With ∼20 particles and both direct and inverse masses, the number of triples is $\binom{20}{3} \times 2 \approx 2300$, and quartets $\binom{20}{4} \approx 4800$. At $|Q - 3/2| < 10^{-3}$ for triples, the expected number of random hits from a flat Q distribution is ∼5. We find ∼3 inverse triples at this level, consistent with random but also consistent with a small number of genuine relations. The four-mass hits at $|Q_4 - 2| < 3 \times 10^{-5}$ are far more significant: with ∼5000 quartets, the random expectation is ∼0.3; we find 2.

15.2 The $\sqrt{m_b} - \sqrt{M_\tau} = 3(\sqrt{m_s} - \sqrt{m_d})$ identity

The two charge sum rules $q_b \approx 3q_s + q_c$ and $q_\tau \approx 3q_d + q_c$ share the same structure and can be subtracted:

$$\boxed{\sqrt{m_b} - \sqrt{M_\tau} = 3(\sqrt{m_s} - \sqrt{m_d})\,, \qquad (0.08\%).} \tag{47}$$

This cross-sector identity links the bottom–tau charge splitting to three times the strange–down charge splitting. In the sBootstrap, b and τ are the heaviest members of their respective $\mathrm{SU}(3)_D$ and lepton sectors; s and d are the lightest down-type turtles. Equation (**??**) says that their charge separations are locked in a 3 : 1 ratio, with the factor 3 suggesting a color multiplicity.

15.3 Internal consistency: two predictions of M_τ

The Koide equation with $\delta = 2/9$ predicts $M_\tau^{(\mathrm{Koide})} = 1776.99$ MeV from M_e alone. The Z-ladder predicts $M_\tau^{(\mathrm{ladder})} = 1776.23$ MeV from m_b and M_Z. These use completely independent physics (lepton angles vs. quark masses and electroweak scale), yet agree to 0.04%. This cross-check provides non-trivial evidence that the Z-ladder and the Koide equation are aspects of the same underlying structure.

15.4 Ranked catalogue

All 19 relationships, ordered by precision:

#	Relationship	Precision	Type
1	$Q_4(t, D_s, \eta_c, \eta_b) = 2$	1.5×10^{-5}	four-mass Koide
2	$Q(e, \mu, \tau) = 3/2$	2×10^{-5}	lepton Koide
3	$Q_4(\pi^0, \pi^\pm, K^\pm, \eta_b) = 2$	2.5×10^{-5}	four-mass Koide
4	$m_c = \sqrt{M_\tau M_Z}/10$	5.6×10^{-5}	Z-ladder
5	$M_t/M_Z = 10^3/23^2$	1.3×10^{-4}	integer ratio
6	$m_{\pi^0} = \sqrt{M_\mu M_Z}/23$	1.5×10^{-4}	meson Z-ladder
7	$\sqrt{m_b} = 3\sqrt{m_s} + \sqrt{m_c}$	1.8×10^{-4}	charge sum
8	$m_s = \sqrt{m_d M_Z}/7$	2.5×10^{-4}	Z-ladder
9	$m_{\eta_c} = \sqrt{M_\tau M_H}/5$	3.4×10^{-4}	H-ladder
10	$V_{us} = \sqrt{m_d/m_s}$	4.3×10^{-4}	Gatto–Sartori–Tonin
11	$m_u = \sqrt{M_e M_Z}/100$	6.3×10^{-4}	Z-ladder
12	$m_b = \sqrt{m_{\pi^\pm} M_H}$	6.7×10^{-4}	geometric mean
13	$K(1/s, 1/\tau, 1/H) = 3/2$	6.8×10^{-4}	inverse Koide
14	$\sqrt{M_\tau} = 3\sqrt{m_d} + \sqrt{m_c}$	7.0×10^{-4}	charge sum
15	$\theta_C = 2/9 + \pi\alpha/6$	7.9×10^{-4}	Cabibbo formula
16	$M_\tau = \sqrt{m_b M_Z}/11$	8.1×10^{-4}	Z-ladder
17	$M_H = \sqrt{M_t M_Z}$	1.5×10^{-3}	geometric mean
18	$K(1/d, 1/s, 1/b)\vert_{M_Z} = 3/2$	1.6×10^{-3}	inverse Koide
19	$q_{\mathrm{boson}} = \sqrt{\pi}\, q_{\mathrm{quarks}}$	2.0×10^{-3}	charge sum

15.5 Physical speculations

1. The Z-ladder formula $y_X^2/y_Y = g_Z\sqrt{2}/(2n^2)$ has tree-level strength. In a model with n Higgs doublets acquiring VEVs in integer ratios, the effective Yukawa matrix $Y_{\mathrm{eff}} = \sum_i Y_i v_i/v$ would produce exactly this structure. The integers $n = 7, 10, 11, 23, 100$ would then encode the VEV landscape.

2. The charge sum rule $\sqrt{m_b} \approx 3\sqrt{m_s} + \sqrt{m_c}$ could reflect a constituent structure where the bottom charge receives one unit from each of the three colours of the strange preon plus one charm preon.

3. The pion connects to the top via $m_{\pi^0}^2/M_\mu = M_t/1000$ (0.018%), which combined with the pion Z-ladder gives $M_t/M_Z = 10^3/23^2$. The lightest hadron encodes the electroweak hierarchy through a single integer.

4. The four-mass $Q_4 = 2$ condition extends the Z_3 preon model to Z_4. The inverse four-mass $K_4(1/u, 1/d, 1/H, 1/\eta_c) = 2$ (2×10^{-5}) is the most precise Koide-type relation found in this analysis.

5. The cross-sector tuple $K(1/s, 1/\tau, 1/H)$ hints at a mass-inversion symmetry in the Yukawa sector, connecting the third-generation turtles to the Higgs.

6. The Cabibbo formula $\theta_C = 2/9 + \pi\alpha/6$ suggests that the CKM mixing angle receives an electromagnetic correction to the Koide angle, with the correction proportional to the exact chain angle $15^\circ = \pi/12$.

7. The integer 23 has a concrete field-theoretic identity: the one-loop QCD beta function coefficient for $N_f = 5$ active flavors is $b_0 = (33 - 2N_f)/3 = 23/3 = n_\pi/k_d$, where $n_\pi = 23$ is the pion Z-ladder integer and $k_d = 3$ is the sBootstrap turtle count. This connects the pion mass ladder directly to the running of α_s.

8. All six Z-ladder integers can be expressed in terms of $N_c = 3$ and $N_f = 5$:

n	Formula	Link
1	trivial	$t \to H$
7	$N_f + N_c - 1$	$d \to s$
10	$2N_f$	$\tau \to c$
11	$2N_f + 1$	$b \to \tau$
23	$11N_c - 2N_f = 3b_0$	$\mu \to \pi^0$
100	$(2N_f)^2$	$e \to u$

The top/Z ratio becomes

$$\frac{M_t}{M_Z} = \frac{(2N_f)^3}{(11N_c - 2N_f)^2}, \tag{48}$$

determined entirely by the sBootstrap turtle structure. Strikingly, this formula **selects** $N_f = 5$ **uniquely**:

N_f	M_t/M_Z	M_t (GeV)	Status
3	0.30	27	too light
4	0.82	75	too light
5	**1.89**	**172.4**	✓
6	3.92	357	too heavy
7	7.60	693	too heavy

This is an *independent* derivation of the sBootstrap $N_f = 5$ result: the closure conditions select $N_f = 5$ from the sfermion spectrum; the Z-ladder selects $N_f = 5$ from the top/Z mass ratio. Both give the same answer.

9. The pion decay constant satisfies $f_\pi \approx \sqrt{m_{\pi^0} M_H/1000}$ (0.16%), connecting the hadronic scale to the Higgs through the same factor $1000 = 10^3$ that appears in M_t/M_Z.

15.6 Higgs quartic from N_c, N_f

The Z-ladder predicts $M_H = \sqrt{M_t M_Z}$, with $M_t/M_Z = (2N_f)^3/(11N_c - 2N_f)^2$. The Higgs quartic coupling $\lambda = M_H^2/(2v^2)$ then evaluates to

$$\lambda = \frac{N_f^3\, g_Z^2}{(11N_c - 2N_f)^2} = \frac{125\, g_Z^2}{529} = 0.1296, \qquad (0.28\%). \tag{49}$$

The Higgs self-coupling is determined by the Z coupling and the sBootstrap integers—no new parameter is needed.

15.7 Kaon mass from M_Z

The kaon mass satisfies

$$m_{K^\pm} = \sqrt{\frac{8\, M_Z}{3}}, \qquad (0.11\%). \tag{50}$$

Since $8/3 = 1/\sin^2\theta_W^{\rm GUT}$ (the tree-level SU(5) prediction $\sin^2\theta_W = 3/8$): $m_K^2 = M_Z/\sin^2\theta_W^{\rm GUT}$. This connects the kaon mass to the GUT-scale electroweak mixing.

15.8 η' as meson geometric mean

The η' mass satisfies

$$m_{\eta'} = \frac{\sqrt{m_{D^0}\cdot m_{D_s}}}{2}, \qquad (0.018\%). \tag{51}$$

The η' is the geometric mean of the two charmed meson masses, divided by $2 = k_u$.

15.9 Pion mass splitting

The Shah–Mir pion mass relation $m_{\pi^\pm}/m_{\pi^0} = 1 + \sqrt{M_\mu/M_Z}$, already noted in the sBootstrap paper [**?**], holds to 7 ppm. Combined with the pion Z-ladder $m_{\pi^0} = \sqrt{M_\mu M_Z}/23$, it gives

$$m_{\pi^\pm} = \frac{\sqrt{M_\mu}\left(\sqrt{M_Z} + \sqrt{M_\mu}\right)}{23}, \tag{52}$$

a formula for the charged pion mass in terms of the muon and Z masses with a single integer.

15.10 SU(5) Casimir identity: $Q_{\rm Koide} = C_2(\mathbf{10})/C_2(\mathbf{5})$

The quadratic Casimir ratio of the antisymmetric two-tensor to the fundamental of SU(N) is $2(N-2)/(N-1)$. Setting this equal to the Koide ratio:

$$\frac{C_2(\wedge^2\mathbf{N})}{C_2(\mathbf{N})} = \frac{2(N-2)}{N-1} = \frac{3}{2} \qquad \Longrightarrow \qquad N = 5. \tag{53}$$

This is the **unique** integer solution, selecting $N = 5 = N_f$ from the Koide ratio alone. Further Casimir identities give Z-ladder integers: $N[C_2(\mathbf{24}) - C_2(\mathbf{10})] = 7$ and $N[C_2(\mathbf{15}) - C_2(\mathbf{10})] = 10$. The inverse discriminant $\Delta_{\rm II} = 9 = C_2(\mathbf{10})\cdot N/2$. Additionally, the Dynkin index ratios $T(\mathbf{15})/T(\mathbf{5}) = 7$ and $T(\mathbf{24})/T(\mathbf{5}) = 10$ give two more Z-ladder integers, while $T(\mathbf{10})/T(\mathbf{5}) = N - 2 = 3 = N_c$. All key constants are simple functions of $N = 5$: $Q = 2(N-2)/(N-1)$,

$n = 7 = N+2$, $n = 10 = 2N$, $N_c = N-2$, $\Delta_{\mathrm{II}} = (N-2)(N+1) = 9$. A **second** Casimir identity gives the Koide angle:

$$\delta = \frac{C_2(\mathrm{fund}_{SU(3)})}{C_2(S^3(\mathrm{fund}_{SU(3)}))} = \frac{4/3}{6} = \frac{2}{9}, \tag{54}$$

using the SU(3) generation group embedded in $G_2 = \mathrm{Aut}(\mathbb{O})$. The symmetric three-tensor $S^3(\mathbf{3})$ has dimension 10 and Casimir 6. Thus Q comes from $\mathrm{SU}(5)_{\text{flavour}}$ Casimirs and δ from $\mathrm{SU}(3)_{\text{generation}}$ Casimirs—the complete mass spectrum from group theory alone. The chain angle $\delta_{scb} \approx 3\delta_\ell = 2/3$ (1° from measured 37.2°) identifies the factor 3 as the colour multiplicity N_c.

15.11 PCAC reformulation of the pion Z-ladder

The pion Z-ladder $m_{\pi^0}^2 = M_\mu M_Z/23^2$ combined with the PCAC relation $m_{\pi^0}^2 f_\pi^2 = (m_u+m_d)\langle\bar{q}q\rangle$ gives

$$(m_u+m_d)\langle\bar{q}q\rangle = \frac{M_\mu \cdot M_Z \cdot f_\pi^2}{23^2}. \tag{55}$$

This relates the QCD chiral condensate to the electroweak masses through the QCD beta function coefficient $23 = 3b_0(N_f=5)$. Numerically, both sides equal 18216 $\mathrm{MeV}^2 \times f_\pi^2$ to 0.03%.

15.12 Open questions

- What mechanism produces the Z-ladder integers? Do they arise from a discrete symmetry, modular forms, or a multi-Higgs VEV landscape?
- Does the Z-ladder survive full SM RGE running? The inverse tuple $K(1/d, 1/s, 1/b)$ improves at M_Z; do the Z-ladder relations also improve at higher scales?
- Can the Cabibbo formula be derived from a modular or radiative mechanism?
- What is the group-theoretic origin of 23? Its double appearance (in both the pion and η_c links) and its connection to M_t/M_Z suggest it is structural, not accidental.
- Is the $\sqrt{\pi}$ boson-quark charge relation a partition-function identity, or a coincidence?

15.13 Testable predictions

If the Z-ladder and charge sum rule are exact, they predict masses more precisely than current measurements:

Mass	Z-ladder prediction	PDG 2024
m_c	1272.93 ± 0.05 MeV	1273.0 ± 2.8 MeV
M_t^{pole}	172.378 ± 0.004 GeV	172.4 ± 0.7 GeV
M_H	125.38 ± 0.26 GeV	125.20 ± 0.11 GeV

The m_c prediction is 50× more precise than the current PDG value. Future lattice QCD determinations or e^+e^- threshold measurements could test it at the required precision. The M_t prediction will be testable at a future lepton collider. The M_H prediction is currently in 0.7σ tension with the PDG value—watchable but not conclusive.

References

[1] S. Antusch, K. Hinze and S. Saad, "Updated Running Quark and Lepton Parameters at Various Scales," arXiv:2510.01312 [hep-ph] (2025).

[2] See e.g. discussions in R. Foot, A. Kobakhidze and R. R. Volkas, Phys. Lett. B **655**, 156 (2007); F. Jegerlehner, Acta Phys. Polon. B **45**, 1167 (2014).