From Newton to the Path Integral

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Abstract

Physically meaningful laws arise as controlled limits of composable local refinements. This paper develops that thesis across classical and quantum theory. We begin with Newton's polygonal approximation of central-force motion and its limit to continuous dynamics, then re-express the same logic in modern variational form through additive action functionals. The path integral is treated as a composition law over refined time slices rather than an isolated quantum postulate, and deformation quantization and renormalization are framed as two mathematically distinct control mechanisms for limit consistency. The narrative is constructive: each stage retains the previous one as a limiting or compatibility condition. Within this architecture we reserve a dedicated role for point-like (Dirac-supported) probes in weak formulations of the action principle, specifying where they are mathematically valid and where regularization is mandatory. The result is a staged program from Newtonian limit methods to quantum amplitudes in which the classical equations are recovered as stationary limits of a broader compositional framework.

1. Introduction

The problem addressed here is not "how to quantize" but how to define a stable continuum theory from iterative refinement. The paper treats Newtonian mechanics, action principles, path integration, deformation quantization, and renormalization as parts of one continuity problem.

The first anchor is Newton's geometric method in central-force motion: replace a curve by a sequence of short segments, impose a local update rule, and pass to a limit while controlling what is meant by "vanishing" quantities. In modern language, the key object is not a smallest geometric piece but a refinement procedure with invariant content [Newton1687]. (The term "uncuttable" in this program means defined only via a refinement limit, not indivisible in the ontological sense.)

The second anchor is the action formulation. Action is additive under temporal partitioning, and that additivity is exactly the algebraic structure needed to compare coarse and fine descriptions. This creates the bridge to quantum composition: if local contributions compose multiplicatively while the underlying functional is additive, exponential weighting is structurally natural [Dirac1933] [Feynman1948].[1]

The foundational tension can be phrased as a refinement paradox: a refinement description is an *infinite-limit* construction, and the limit is not automatically unique or even defined once the refined objects become singular (Dirac-supported probes, distributional derivatives) or once intermediate

[1]The additivity of the action under temporal partition forces the Lagrangian to be local, up to boundary terms: cross-temporal correlations would break $S[\gamma_{if}] = S[\gamma_{im}] + S[\gamma_{mf}]$ for generic paths, so no cross-time kernel survives. Locality of L is a Stage 1 forcing result, though weaker than the composition-forcing at Stage 2 — it constrains form, not content. With Ostrogradsky stability, $L = L(t, q, \dot{q})$ follows.

quantities diverge. Berkeley's critique of Newton's fluxions — the "ghosts of departed quantities" objection [Berkeley1734Analyst] — is an early articulation of this tension. Classical mechanics often hides it by treating "send the refinement parameter to zero" as an axiomatically legitimate operation. The program pursued here instead keeps refinement explicit, isolates where limit-taking needs extra control, and treats **quantization** and **renormalization** as two distinct mechanisms for making refined composition stable when the naive Newtonian limit is not rigorous as written.

Three recurring obstructions make "refine $\to 0$" nontrivial in practice:

1. **Singular probes:** point-supported variations and corners/impulses force distributional weak forms (mollifiers and contact terms).
2. **Non-uniqueness:** refinement/composition can admit multiple classically equivalent but quantum-distinct schemes (ordering/discretization choices), requiring an explicit equivalence or control map. (Minimal witness: time-slicing $H = pq$ can yield $-\hat{p}\hat{q}$ vs $-\hat{q}\hat{p}$, differing by $O(\hbar)$ as operators; requiring symmetry of the generator — the minimal condition for self-adjointness, needed for unitarity — selects the midpoint (half-density) convention.)
3. **Divergence:** some refinement limits do not converge without subtraction/parameter flow (renormalization). (Toy witness: the derivative exists only after subtracting a $1/\varepsilon$ divergence in the difference quotient.)

This manuscript treats these as limit-control problems rather than as postulates about "nature at the smallest scale."

Remark (No Lebesgue measure on path space). In infinite-dimensional spaces there is no nontrivial translation-invariant σ-finite Borel measure (no Lebesgue/Haar measure) [Sudakov1959] [Glimm-Jaffe1987] (see also [Velhinho2017InfDimMeasure] for a modern survey), so the formal symbol Dq in a path integral cannot be interpreted as an ordinary "Lebesgue measure on trajectories." The standard prescription "refine the time slicing, integrate over paths, and send $\Delta t \to 0$" is therefore not a raw Newtonian limit statement but a definition-by-refinement that must specify normalization and, when singularities are present, regulator/subtraction rules.

Remark (Constants as compatibility-limit parameters). In this program, $\hbar$, c, and G are read as control parameters for three distinct compatibilities: $\hbar \to 0$ recovers classical stationarity from oscillatory composition, $c \to \infty$ recovers Galilean/Newtonian kinematics from Lorentz-compatible refinement, and $G \to 0$ switches off geometric backreaction (with $\hbar, c, G$ together defining the Planck scale where quantum and gravitational refinement controls meet). A concrete example of a $c \to \infty$ passage requiring an explicit subtraction convention is given in Section 2.4.

The third anchor is methodological. Deformation quantization and renormalization are two ways to control limits: 1. Deformation quantization controls the classical-to-quantum passage through algebraic deformation and recovery of Poisson structure in the small-parameter limit [Landsman1998] [Connes1994]. 2. Renormalization controls divergent refinement procedures by regulator-dependent intermediate steps and regulator-independent observables [ConnesKreimer2000].

Section 2 fixes the formal vocabulary and claim taxonomy. The paper does not assume that continuum limits are ontological statements about nature — only that they are operational definitions of stable predictive objects. This assumption will be stress-tested in later sections.

Contributions. 1. A refinement/composition reading of the Newton $\to$ action $\to$ kernel chain in which each stage is retained as a compatibility condition, not replaced. 2. An intrinsic half-density formulation of the composition law for propagators, separating coordinate-free kernel composition

from scalarization conventions. 3. A semigroup-closure derivation showing the short-time normalization exponent $t^{-d/2}$ is forced by composition (the "square-root Jacobian"). 4. A necessity theorem (Proposition 6.1) showing that an action-dimensional scale $\kappa = \hbar$ is forced by composition alone, with the identity limit and dimensional homogeneity derivable from composition combined with the physical setup (action-based dynamics on $\mathbb{R}^d$ with mass m). The exponential weight form is derived rather than assumed; extensions to curved, interacting, and Lorentzian settings are discussed. 5. A refinement-compatibility framing of renormalization in which RG invariance is the consistency condition demanded by divergent refinement limits. 6. A fully explicit "RG appears before QFT" computation (2D delta/contact interaction) included as an appendix-level witness.

2. Notation and Claim Taxonomy

Dimension bookkeeping. Throughout Sections 2–7, d denotes the dimension of the manifold being integrated over in the composition law (typically configuration-space/spatial dimension in nonrelativistic kernels). When field-theory-style spacetime integrals appear, spacetime dimension is denoted D to avoid conflating it with the composition-variable dimension.

2.1 Core Objects

Let $q : [t_i, t_f] \to \mathbb{R}^d$ be a configuration-space trajectory and $\mathcal{L}(q, \dot{q}, t)$ a Lagrangian. Define the action:

$$S[q] = \int_{t_i}^{t_f} \mathcal{L}(q, \dot{q}, t)\, dt.$$

For a partition $t_i = t_0 < t_1 < \cdots < t_N = t_f$ with $\Delta t_k = t_{k+1} - t_k$, define the discrete action functional:

$$S_N[q] = \sum_{k=0}^{N-1} \mathcal{L}\left(q_k, \frac{q_{k+1} - q_k}{\Delta t_k}, t_k\right) \Delta t_k.$$

For planar central motion $q = (r, \theta)$, define areal velocity and angular momentum:

$$\dot{A} = \frac{1}{2} r^2 \dot{\theta}, \qquad L_{\text{ang}} = m r^2 \dot{\theta} = 2m\dot{A}.$$

These definitions serve as the Newtonian-to-variational bridge in Sections 3 and 4.

2.2 Weak-Form Preliminaries for Point-Like Probes

Let $\eta \in C_c^\infty((t_i, t_f); \mathbb{R}^d)$ be a smooth compactly supported test variation. The first variation is written $\delta S[q; \eta]$, and stationarity means $\delta S[q; \eta] = 0$ for all admissible η.

To model point-supported probes later, introduce a mollifier family ρ_ε with $\rho_\varepsilon \rightharpoonup \delta$ in distributions as $\varepsilon \to 0^+$. Any use of Dirac-supported variations in this manuscript is understood as a mollified limit unless explicitly labeled heuristic.

2.3 Claim Taxonomy

Every substantive claim is marked by one of: 1. **Proposition**: statement intended as mathematically valid under explicit assumptions. 2. **Derivation**: explicit calculation from stated premises. 3. **Heuristic**: physically motivated bridge that is not presented as full proof.

2.4 Seed Claims

Proposition 2.1 (Additive refinement structure). Given a partition of $[t_i, t_f]$, the discrete action S_N is additive over concatenated subintervals by construction. Action is therefore a natural candidate for refinement comparison.

Derivation 2.1 (Composition-compatible exponential form). Suppose a weight map W on time-sliced paths satisfies: 1. $W[\gamma_1 \circ \gamma_2] = W[\gamma_1]W[\gamma_2]$ for composable segments. 2. $\log W$ is local in the slice contributions. 3. The corresponding additive functional is proportional to S_N in the refinement limit.

Then there exists a scale κ and constant c_0 such that, up to normalization, $W[\gamma] \propto \exp(c_0 S[\gamma]/\kappa)$. Choosing $c_0 = i$ and $\kappa = \hbar$ recovers the standard oscillatory quantum weighting form.

Heuristic 2.1 (Classical recovery as concentration). When the phase scale is small relative to action variation, dominant contributions concentrate near stationary-action trajectories. This is the structural claim later made precise through stationary-phase analysis.

Derivation 2.2 (Nonrelativistic limit as a controlled $c \to \infty$ subtraction). Consider the relativistic free-particle action written with an explicit speed-of-light parameter:

$$S_{\mathrm{rel}}[q] = -mc^2 \int_{t_i}^{t_f} dt \, \sqrt{1 - \frac{\|\dot{q}(t)\|^2}{c^2}}.$$

For $\|\dot{q}\| \ll c$, expand the square root:

$$S_{\mathrm{rel}}[q] = \int_{t_i}^{t_f} dt \left(-mc^2 + \frac{1}{2} m \|\dot{q}(t)\|^2 + O(c^{-2}) \right).$$

The term $-mc^2(t_f - t_i)$ diverges as $c \to \infty$. Classically it is inert (adding a constant to $\mathcal{L}$ does not change the Euler-Lagrange equations), so one may subtract it as an allowed additive counterterm to obtain a finite $c \to \infty$ limit: the Newtonian kinetic action $\int \frac{1}{2} m \|\dot{q}\|^2 dt$ plus higher-order relativistic corrections. In quantum amplitudes, the same subtraction corresponds to an overall phase $e^{-imc^2(t_f - t_i)/\hbar}$. This calculation is at the particle-mechanics level; field-theory and gravity effects of constant terms (vacuum energy) are a separate issue not addressed here.

2.5 Scope Boundary

This section fixes notation and methodological boundaries: 1. Historical statements are used only as source-anchored motivation. 2. Mathematical validity requires explicit assumptions and, for singular objects, explicit regularization. 3. Quantum and QFT-level statements are introduced only after the composition law and refinement language are fixed.

> **Reader map (compatibilities).** - **Partition compatibility** ($\mathcal{C}_t$): temporal refinement/composition (time slicing). See Sections 3–4. - **Representation compatibility** ($\mathcal{Q}_\hbar$): ordering/discretization choices with the same $\hbar \to 0$ limit. See Sections 6–7.

- **Scale compatibility** ($\mathcal{R}_\Lambda$): coarse/fine scale comparison after parameter running (RG). See Section 8 (and Appendix 10.5 for an explicit witness).

Symbol definitions and formal summary: Appendix 10.3.

3. Newtonian Refinement and Area Law

3.1 Source-Critical Framing

In Book I, Proposition I of the *Principia*, Newton proves that a centripetal forcing rule implies equal areas swept in equal times by the radius vector. The historical proof is polygonal and limit-based: one constructs a piecewise-linear trajectory with impulses directed to a fixed center, then passes to a continuous curve by refinement [Newton1687].

This section uses that structure directly and then translates to modern vector notation. The statements below distinguish: 1. Newton's geometric argument about polygons and limits. 2. A modern reformulation via torque and angular momentum.

The reformulation is mathematically equivalent under the same assumptions, but it is an interpretive translation, not a verbatim historical rendering.

3.2 Discrete Refinement Model

Fix equal time steps $\Delta t > 0$, times $t_k = t_0 + k\Delta t$, and a fixed center O. Let $\mathbf{r}_k$ be the position vector at t_k. The stepwise model is: 1. Free inertial drift between t_k and t_{k+1}. 2. Instantaneous impulse at each vertex t_k, directed along $\mathbf{r}_k$ (centripetal/central).

Let $\mathbf{v}_k^-$ be velocity just before the impulse at t_k, and $\mathbf{v}_k^+$ just after. The impulse condition is

$$m(\mathbf{v}_k^+ - \mathbf{v}_k^-) = J_k\,\hat{\mathbf{r}}_k, \qquad \hat{\mathbf{r}}_k = \frac{\mathbf{r}_k}{\|\mathbf{r}_k\|}.$$

Drift then gives

$$\mathbf{r}_{k+1} = \mathbf{r}_k + \mathbf{v}_k^+\Delta t, \qquad \mathbf{v}_{k+1}^- = \mathbf{v}_k^+.$$

Derivation 3.1 (Finite-step angular momentum invariance). Define $\mathbf{L}_k^- = m\,\mathbf{r}_k \times \mathbf{v}_k^-$, $\mathbf{L}_k^+ = m\,\mathbf{r}_k \times \mathbf{v}_k^+$.

At impulse:

$$\mathbf{L}_k^+ - \mathbf{L}_k^- = m\,\mathbf{r}_k \times (\mathbf{v}_k^+ - \mathbf{v}_k^-) = \mathbf{r}_k \times (J_k\hat{\mathbf{r}}_k) = \mathbf{0}.$$

During drift:

$$\mathbf{L}_{k+1}^- = m\,\mathbf{r}_{k+1} \times \mathbf{v}_{k+1}^- = m(\mathbf{r}_k + \mathbf{v}_k^+\Delta t) \times \mathbf{v}_k^+ = m\,\mathbf{r}_k \times \mathbf{v}_k^+ = \mathbf{L}_k^+.$$

Hence $\mathbf{L}_{k+1}^- = \mathbf{L}_k^-$: angular momentum is exactly conserved at every finite step in this refinement model.

Derivation 3.2 (Equal areas in equal times, discrete form). The swept area in step k is the triangle area

$$\Delta A_k = \frac{1}{2}\left\|\mathbf{r}_k \times (\mathbf{r}_{k+1} - \mathbf{r}_k)\right\| = \frac{1}{2}\left\|\mathbf{r}_k \times \mathbf{v}_k^+\right\| \Delta t = \frac{\|\mathbf{L}\|}{2m}\Delta t.$$

Therefore for fixed Δt, ΔA_k is independent of k. This is the equal-areas statement at finite polygonal level.

Derivation 3.3 (Numerical witness for the finite-step area law). Consider an inverse-square force with $GM = m = 1$. Place a body at $\mathbf{r}_0 = (1, 0)$ with velocity $\mathbf{v}_0 = (0, 0.8)$ — this is apoapsis of an ellipse with eccentricity $e = 9/25 = 0.36$ and semi-major axis $a \approx 0.735$. The angular momentum is $L = 0.8$.

Running the kick-drift scheme of Section 3.2 with $N = 12$ equal steps over one period ($T \approx 3.96$): every triangular area ΔA_k equals $L\,\Delta t/(2m) \approx 0.132$ exactly in the algebraic sense of Derivations 3.1–3.2, despite the radial distance varying by a factor $r_{\text{apo}}/r_{\text{peri}} = (1+e)/(1-e) = 17/8 \approx 2.1$ between apoapsis and periapsis. The equality is not a numerical coincidence or a continuum approximation — it is an algebraic identity at each finite step, holding for any central force. Only the trajectory shape converges under refinement; the swept-area invariant is exact at every N.

3.3 Continuum Passage and Central-Force Generality

Proposition 3.1 (Refinement limit of areal velocity). If $\max_k \Delta t_k \to 0$ under consistent refinement, the finite-step law above yields

$$\frac{dA}{dt} = \frac{\|\mathbf{L}\|}{2m},$$

for the limiting trajectory whenever the limit exists in the standard differentiable sense.

For a smooth central force $\mathbf{F}(\mathbf{r}) = f(r)\hat{\mathbf{r}}$, this same invariant follows from torque:

$$\frac{d\mathbf{L}}{dt} = \mathbf{r} \times \mathbf{F} = \mathbf{0}.$$

The areal law is therefore independent of the inverse-power index n in $\mathbf{F} = -(K/r^n)\hat{\mathbf{r}}$: n changes radial dynamics and orbit families, but not the areal-velocity conservation mechanism itself. A companion note on relativistic central orbits analyzes Lorentz-compatible kinematics in detail.

Remark (Asymptotic threshold: Newton vs SR). For attractive power-law $F(r) = K/r^q$, fixed angular momentum L, and high-momentum kinetic asymptotic $E_{\text{kin}}(p) \sim ap^\nu$, one has $p \sim L/r$ as $r \to 0$. The centrifugal barrier then scales as $r^{-\nu}$ while the attraction scales as $r^{-(q-1)}$, giving threshold $q_{\text{crit}} = \nu + 1$. This recovers the Newtonian/SR shift ($\nu = 2 \Rightarrow q_{\text{crit}} = 3$, $\nu = 1 \Rightarrow q_{\text{crit}} = 2$) in one line. The criterion is a fixed-L, small-r asymptotic statement and is distinct from the separate inverse-square dimensional identity mechanism (where K/L has velocity units and yields $v = K/L$ in SR circular analysis).

Remark (Impulse-to-continuous interpretation). The impulse model is a refinement scaffold for continuous forcing, not a literal claim that nature acts by discrete kicks. Its value is structural: invariants proven exactly at finite step survive controlled refinement. The passage from discrete

polygonal orbits to a continuous trajectory depends on Newton's Lemma 3 (Book I, Section 1 of the *Principia*) and has been the subject of scholarly debate. Nauenberg [Nauenberg2003KeplerArea] gives a modern reconstruction showing the polygonal construction has a well-defined continuum limit parameterizing a continuous planar curve; Pourciau [Pourciau2003] critically analyzes the same limit and identifies conditions under which the impulse assumption requires additional care. The hedge in Proposition 3.1 ("whenever the limit exists in the standard differentiable sense") is sufficient for the present purposes: the structural content of the equal-area invariant at finite step is independent of the convergence subtleties.

Remark (Newton's scheme as a symplectic integrator). The kick-drift structure of Section 3.2 — update velocity by an impulse at the current position, then drift — is precisely the symplectic Euler integrator applied to the separable Hamiltonian $H(\mathbf{r}, \mathbf{p}) = |\mathbf{p}|^2/(2m) + V(r)$ [Nauenberg1994SymplecticNewton] [Nauenberg2018GraphicalMethod]. The exact angular-momentum conservation (Derivation 3.1) follows from two properties: the force is central (so each kick preserves $\mathbf{L}$), and the free drift is linear (so it also preserves $\mathbf{L}$). More broadly, the symplectic character of the map means it preserves the canonical 2-form $\omega = d\mathbf{p} \wedge d\mathbf{r}$ at each finite step. Standard convergence theory for one-step numerical integrators guarantees that the polygonal orbit converges to the true continuous orbit with global error $O(h)$ on any finite time interval where the force ∇V is Lipschitz — i.e. away from the $r = 0$ collision singularity of central potentials. At the collision point the Lipschitz constant diverges, the fixed-step scheme breaks down, and regularization (Kustaanheimo-Stiefel / Levi-Civita coordinate transformations) or adaptive stepping is required. The structural invariants (angular momentum, equal areas) are exact at each finite step regardless and do not depend on the convergence of the interpolation limit. While the method has the same first-order convergence rate as the explicit Euler scheme analyzed in Section 8.4, its step-doubling corrections are constrained to preserve the symplectic form — a finite-step requirement analogous to the principle that renormalization counterterms must respect the symmetries of the theory.

3.4 Closing the Area-Law Question

Section 2 left one key ambiguity open: is Newton's area law a small-step approximation or a genuine invariant statement? The derivations above close that point: within the polygonal central-impulse model, the equal-area law is exact at each finite step and only the curve interpolation is a limiting passage.

4. Action as Additive Invariant

4.1 Stationarity Setup

The Section 3 invariant was derived from a refinement model in configuration geometry. We now restate the same physics through stationarity of action.

Assume: 1. $q : [t_i, t_f] \to \mathbb{R}^d$ is at least C^2, and variations η are C^1 (or smooth with compact support). 2. $\mathcal{L}(q, \dot{q}, t)$ is C^1 in t and C^2 in $(q, \dot{q})$ on the region reached by $(q(t), \dot{q}(t))$.

Let the action be

$$S[q] = \int_{t_i}^{t_f} \mathcal{L}(q(t), \dot{q}(t), t)\, dt,$$

and define $q_\varepsilon = q + \varepsilon\eta$ for an admissible variation η, with either: 1. fixed endpoints $\eta(t_i) = \eta(t_f) = 0$, or 2. compact support in (t_i, t_f).

Stationarity means

$$\delta S[q;\eta] = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} S[q_\varepsilon] = 0 \quad \text{for all admissible } \eta.$$

Proposition 4.1 (Fundamental lemma, vector form). If $F : [t_i, t_f] \to \mathbb{R}^d$ is continuous and $\int_{t_i}^{t_f} F(t) \cdot \eta(t)\, dt = 0$ for all $\eta \in C_c^\infty((t_i, t_f); \mathbb{R}^d)$, then $F(t) = 0$ for all $t \in (t_i, t_f)$.

4.2 Euler-Lagrange Derivation

Derivation 4.1 (Euler-Lagrange equation). Differentiate under the integral sign (justified by the smoothness assumptions). By the chain rule, $\frac{d}{d\varepsilon}\big|_0 \mathcal{L}(q + \varepsilon\eta, \dot{q} + \varepsilon\dot{\eta}, t) = \frac{\partial \mathcal{L}}{\partial q} \cdot \eta + \frac{\partial \mathcal{L}}{\partial \dot{q}} \cdot \dot{\eta}$. Therefore:

$$\delta S[q;\eta] = \int_{t_i}^{t_f} \left(\frac{\partial \mathcal{L}}{\partial q} \cdot \eta + \frac{\partial \mathcal{L}}{\partial \dot{q}} \cdot \dot{\eta} \right) dt.$$

Integrating the second term by parts:

$$\delta S[q;\eta] = \left[\frac{\partial \mathcal{L}}{\partial \dot{q}} \cdot \eta \right]_{t_i}^{t_f} + \int_{t_i}^{t_f} \left(\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} \right) \cdot \eta\, dt.$$

Endpoint or compact-support conditions remove the boundary term. By Proposition 4.1, stationarity for all admissible η implies:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = 0.$$

This is the Euler-Lagrange equation.

4.3 Rotational Symmetry and Angular Momentum

For planar central motion with

$$\mathcal{L}(r, \theta, \dot{r}, \dot{\theta}) = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - V(r),$$

θ is cyclic ($\partial \mathcal{L}/\partial \theta = 0$). Applying Euler-Lagrange to θ gives:

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2\dot{\theta} = L_{\text{ang}} \quad \Rightarrow \quad \frac{dL_{\text{ang}}}{dt} = 0,$$

which is the rotational Noether conservation law [Noether1918].

In full vector form for $\mathcal{L}(\mathbf{r}, \dot{\mathbf{r}}) = \frac{m}{2}\|\dot{\mathbf{r}}\|^2 - V(\|\mathbf{r}\|)$, the canonical momentum is $\mathbf{p} = \partial \mathcal{L}/\partial \dot{\mathbf{r}} = m\dot{\mathbf{r}}$ and rotational invariance yields the conserved angular momentum vector

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}.$$

Proposition 4.2 (Geometric-variational invariant equivalence). Under the regularity assumptions above, the Section 3 area-law invariant and the Noether charge are the same quantity in different language:

$$\dot{A} = \frac{1}{2} r^2 \dot{\theta} = \frac{L_{\text{ang}}}{2m}.$$

Sections 3 and 4 therefore do not provide competing derivations; they provide geometric and variational presentations of one conserved structure. Newton's polygonal proof is strictly stronger in regularity assumptions: the equal-area statement is exact at each finite impulse step without assuming smoothness or even a Lagrangian, whereas Noether's charge conservation requires a smooth trajectory and rotational symmetry of $\mathcal{L}$ to reproduce the same invariant. (The comparison concerns regularity, not generality: Noether's theorem applies to arbitrary continuous symmetries, not just rotational invariance.)

Proposition 4.3 (Energy for autonomous central motion). If $\mathcal{L}$ has no explicit time dependence, then the energy function

$$E = \dot{q} \cdot \frac{\partial \mathcal{L}}{\partial \dot{q}} - \mathcal{L}$$

is conserved (time-translation symmetry, another Noether law) [Noether1918]. For the central-motion Lagrangian above,

$$E = \frac{m}{2} \dot{r}^2 + \frac{L_{\text{ang}}^2}{2mr^2} + V(r),$$

showing the standard reduction to one-dimensional radial motion with effective potential $V_{\text{eff}}(r) = V(r) + L_{\text{ang}}^2/(2mr^2)$.

4.4 Additivity and Composition Pre-Bridge

Recall the discrete action functional from the refinement viewpoint:

$$S_N[q] = \sum_{k=0}^{N-1} \mathcal{L}\left(q_k, \frac{q_{k+1} - q_k}{\Delta t_k}, t_k\right) \Delta t_k.$$

It is additive under interval concatenation by construction. This additivity is the structural input used later for composition-based quantum weighting in Section 6.

Remark (Toward distributional probes). Point-like probes of extrema can be expressed in distributional language. Technical use of such probes is deferred to Section 5, where mollifier limits and admissibility are stated explicitly.

5. Dirac Distributions and Extremal Action

5.1 Why Weak Formulations Appear Here

The preceding sections treated trajectories as classically smooth. Two themes force a more careful formulation:

1. Refinement limits often produce objects that are only piecewise smooth (corners) or are best handled by weak limits.
2. Point-like probes — Dirac-supported localization — are naturally stated in distribution theory.

Throughout this section, distributions are used in the narrow, standard sense: as linear functionals on test functions and as limits of smooth approximations. Nonlinear operations on distributions are not assumed unless explicitly regularized.

5.2 Weak Euler-Lagrange Equation

Let $q \in C^1([t_i, t_f]; \mathbb{R}^d)$ be a candidate trajectory and assume $\mathcal{L}(q, \dot{q}, t)$ is smooth enough that $\partial_q \mathcal{L}$ and $\partial_{\dot{q}} \mathcal{L}$ are well-defined along q.

Proposition 5.1 (Weak stationarity). If $\delta S[q; \eta] = 0$ for all $\eta \in C_c^\infty((t_i, t_f); \mathbb{R}^d)$, then the Euler-Lagrange operator

$$F[q](t) \equiv \frac{\partial \mathcal{L}}{\partial q}(q, \dot{q}, t) - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}}(q, \dot{q}, t)$$

vanishes as a distribution on (t_i, t_f): for all test η,

$$\int_{t_i}^{t_f} F[q](t) \cdot \eta(t)\, dt = 0.$$

Equivalently, $F[q] = 0$ in $\mathcal{D}'((t_i, t_f); \mathbb{R}^d)$, where $\mathcal{D}'$ is the dual of C_c^∞.

Derivation (Weak form from first variation). Start from the first-variation identity (as in Section 4):

$$\delta S[q; \eta] = \int_{t_i}^{t_f} \left(\frac{\partial \mathcal{L}}{\partial q} \cdot \eta + \frac{\partial \mathcal{L}}{\partial \dot{q}} \cdot \dot{\eta} \right) dt.$$

Integrate the second term by parts. Compact support eliminates the boundary term and yields the stated distributional identity.

5.3 Point-Like Probes via Mollifiers (Not Raw Deltas)

Pick a nonnegative mollifier $\rho \in C_c^\infty(\mathbb{R})$ with $\int \rho = 1$, and define $\rho_\varepsilon(t) = \varepsilon^{-1}\rho(t/\varepsilon)$.

Proposition 5.2 (Localized probing under continuity). Assume $F[q](t)$ is continuous at a time $t_0 \in (t_i, t_f)$. Then weak stationarity implies the pointwise condition $F[q](t_0) = 0$.

Derivation. For any fixed vector $u \in \mathbb{R}^d$, choose a localized test function $\eta_\varepsilon(t) = \rho_\varepsilon(t - t_0)\, u$. Then the weak identity gives

$$0 = \int_{t_i}^{t_f} F[q](t) \cdot \rho_\varepsilon(t - t_0)\, u \, dt = u \cdot \int_{t_i}^{t_f} \rho_\varepsilon(t - t_0)\, F[q](t)\, dt.$$

As $\varepsilon \to 0^+$, the convolution integral tends to $F[q](t_0)$ by continuity. Since u was arbitrary, $F[q](t_0) = 0$.

This is the precise sense in which "Dirac-supported probes" recover pointwise Euler-Lagrange equations: they do so through mollifier limits, not by inserting nonlinear expressions involving $\delta(t - t_0)$. (For an expanded treatment with explicit functional-analytic hypotheses, a fully worked delta-kick model, and the connection from N-impulse matching to the time-sliced path integral, cf. Theorem 2.1 and Section 4 in the Dirac Probes companion note.)

5.4 Corners and Impulses: Jump Conditions

Two distinct phenomena produce "singular" behavior in time:

1. **Corners**: q is continuous but $\dot{q}$ has a jump at t_0, with no delta forcing.
2. **Impulses**: the dynamics includes a delta force at t_0, producing a momentum jump.

Proposition 5.3 (Corner condition without impulse). Assume q is piecewise C^2 with a velocity discontinuity at t_0, and satisfies the unforced Euler-Lagrange equation on each side of t_0. Then:

$$\left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\right]_{t_0^-}^{t_0^+} = 0.$$

Derivation (Corner condition). Integrate the unforced Euler-Lagrange equation on $[t_0 - \varepsilon, t_0 + \varepsilon]$:

$$\left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\right]_{t_0-\varepsilon}^{t_0+\varepsilon} = \int_{t_0-\varepsilon}^{t_0+\varepsilon} \frac{\partial \mathcal{L}}{\partial q}\, dt.$$

Let $\varepsilon \to 0^+$. Under local boundedness of $\partial_q \mathcal{L}$, the right-hand side vanishes, yielding the jump condition above. This is the Weierstrass-Erdmann corner condition: canonical momentum is continuous across velocity discontinuities.

Proposition 5.4 (Impulse force implies momentum jump). Consider the forced Euler-Lagrange equation in distribution form

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = J\, \delta(t - t_0),$$

for a fixed impulse vector $J \in \mathbb{R}^d$. If $\partial_{\dot{q}} \mathcal{L}$ has one-sided limits at t_0, then

$$\left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\right]_{t_0^-}^{t_0^+} \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}}(t_0^+) - \frac{\partial \mathcal{L}}{\partial \dot{q}}(t_0^-) = J.$$

Derivation. Integrate the equation on $[t_0 - \varepsilon, t_0 + \varepsilon]$. The integral of the smooth term $\partial_q \mathcal{L}$ tends to 0 as $\varepsilon \to 0$. The derivative term integrates to the jump in $\partial_{\dot{q}} \mathcal{L}$. The right-hand side integrates to J.

For the standard mechanical Lagrangian $\mathcal{L} = \frac{m}{2}\|\dot{q}\|^2 - V(q)$, this reduces to the familiar momentum jump:

$$m(\dot{q}(t_0^+) - \dot{q}(t_0^-)) = J.$$

This connects directly to the Section 3 impulse scaffold: central impulses preserve angular momentum because they change momentum only in the radial direction.

5.5 Extremal Measures: Finite-Dimensional Analogy and Limits

The phrase "Dirac distributions to calculate extrema" is unambiguous in finite dimensions. For a smooth $f : \mathbb{R} \to \mathbb{R}$, the distribution $\delta(f'(x))$ is supported on the critical points of f. In higher dimensions one analogously uses $\delta(\nabla f)$.

Derivation (Square-root delta normalization and Born-rule form). Let $f : \mathbb{R}^N \to \mathbb{R}$ be smooth and define, for $\varepsilon > 0$, $A_\varepsilon(O) := \varepsilon^{-N/2} \int e^{\frac{i}{\varepsilon} f(x)} O(x)\, dx$. Then $|A_\varepsilon(O)|^2 = \varepsilon^{-N} \iint e^{\frac{i}{\varepsilon}(f(x)-f(y))} O(x)\overline{O(y)}\, dx\, dy$. Under the near-diagonal scaling $y = x + \varepsilon z$ (so $dy = \varepsilon^N dz$), one formally obtains $|A_\varepsilon(O)|^2 \to (2\pi)^N \int \delta(\nabla f(x))\, |O(x)|^2\, dx$. This exhibits the pattern "density = |amplitude|^2", with the exponent $N/2$ matching the half-density scaling needed to cancel Jacobians under refinement.

Derivation (Nondegenerate critical points: why the weights are square roots). In one dimension, if f has finitely many nondegenerate critical points x_i (so $f'(x_i) = 0$ and $f''(x_i) \neq 0$), then the distributional identity

$$\delta(f'(x)) = \sum_i \frac{\delta(x - x_i)}{|f''(x_i)|}$$

makes explicit that $\delta(f')\, dx$ is a density supported on stationary points with weights $1/|f''|$. Stationary phase, by contrast, gives amplitude contributions weighted by $1/\sqrt{|f''(x_i)|}$. This is the clean finite-dimensional reason the "half-density first" viewpoint is natural: amplitude weights are square roots of density weights.

Section 6 recovers the same "square-root Jacobian" in the dynamical setting: semigroup composition of short-time kernels forces the characteristic $t^{-d/2}$ normalization (Derivation in Section 6.2). A comprehensive treatment showing this exponent is forced across four distinct settings — temporal composition, Van Vleck determinant in curved space, heat-kernel diffusion, and renormalization thresholds — is given in the companion satellite [PathIntegralNormalization].

In infinite-dimensional settings (paths), one is tempted to write "formal measures" supported on stationary-action trajectories. In this manuscript such expressions are treated as heuristics until they are regularized and made compatible with composition (Section 6); see also the discussion of Lebesgue measure on path space in Section 1.

Remark (Delta-object dictionary and safe/unsafe criterion). Five delta-adjacent constructions appear in this work: (i) the stationary-set delta $\delta(\nabla f)$, which localizes on critical points of a smooth function with Hessian-determinant weights (the distributional change-of-variables formula); (ii) the distributional derivative δ', defined by duality and realizable as a point-splitting limit; (iii) the

diagonal delta kernel $\delta(x-y)|dx|^{1/2}|dy|^{1/2}$, the Schwartz kernel of the identity on half-densities; (iv) the delta potential $g\,\delta(x)$, a rank-one perturbation that defines a self-adjoint extension in $d \le 3$ but requires renormalization for $d \ge 2$; and (v) the heuristic "$\delta(\delta S)$" for extremal support on path space. The first four are rigorous (with renormalization where noted); the fifth is meaningful only as a shorthand for the finite-dimensional stationary-phase identity combined with the composition-compatible $N \to \infty$ limit of Section 6.

The safe/unsafe boundary is mollifier universality: a delta construction is scheme-independent precisely when it probes a continuous object, so that the mollifier-sequence limit is unique. Products of singular distributions or coincident-point evaluations of singular kernels are scheme-dependent and require explicit regularization.

5.6 Caveats (Nonlinear Distribution Pitfalls)

Three restrictions must be observed:

1. Products like $\delta(t)^2$ are not defined in standard distribution theory; any appearance requires a regularization scheme and a proof of scheme-independence for claimed observables.
2. "Evaluate at a point" is only legitimate for quantities known to be continuous (or otherwise well-defined) at that point; mollifier probing must state this assumption explicitly.
3. Stationarity ($\delta S = 0$) is not the same as minimality; second variation and convexity conditions are separate and are not assumed here.

6. Composition and Path Integral

6.1 Composition Postulate for Amplitudes

Let $K(q_f, t_f; q_i, t_i)$ denote the transition amplitude. The structural postulate is composition on intermediate time slices:

$$K(q_f, t_f; q_i, t_i) = \int dq\, K(q_f, t_f; q, t)\, K(q, t; q_i, t_i), \qquad t_i < t < t_f.$$

Remark (Half-density kernels make composition measure-free). On a configuration manifold M, the coordinate-free object underlying the displayed formula is a **bi-half-density kernel**: $K_t(q', q) \in |\Lambda^d T^*_{q'} M|^{1/2} \otimes |\Lambda^d T^*_q M|^{1/2}$. Composition is then the canonical pairing in the intermediate variable q, requiring no background measure. Writing $\int dq$ is what one gets after choosing a reference density to identify half-densities with scalar functions.

Derivation (Coordinate invariance of composition via half-densities). In local coordinates q, write $K_t(q', q) = k_t(q', q)\,|dq'|^{1/2}|dq|^{1/2}$. Then $K_{t_f - t}(q_f, q)\, K_{t - t_i}(q, q_i) = k_1 k_2\, |dq_f|^{1/2}|dq|\,|dq_i|^{1/2}$ is a density in q, so $\int_M K_{t_f - t}(q_f, q) K_{t - t_i}(q, q_i)$ is coordinate invariant. This is the intrinsic meaning of the composition postulate.

Remark (Scalarization gauge and scale). Writing a half-density kernel as an ordinary scalar function with an explicit "dq" implicitly chooses a reference density ρ_* on M (equivalently a reference half-density $\sigma_* = \rho_*^{1/2}$). Different choices are related by pointwise multiplication and give unitarily equivalent scalar representations. If one additionally demands scalar amplitudes be dimensionless, then σ_* must carry the full length$^{d/2}$ weight, so a universal choice of σ_* is equivalent to choosing a universal length$^{d/2}$ scale. In a spacetime QFT setting where the scalarization problem is formulated

over an integration variable of dimension D, this is a length$^{D/2}$ scale; in $D = 4$ it is an area, with the Planck area $\ell_P^2 = \hbar G/c^3$ a natural universal candidate. A companion note explores further (optional) hypotheses about such scale suppliers; no such identification is required for the present paper's structural chain.

Remark (Kinetic operator simplicity selects $D = 4$ independently of scale arguments). In a covariant/QFT setting (spacetime dimension D), the scalar Laplacian Δ_g induces an operator on half-densities by conjugation,

$$\widetilde{\Delta}_g := |g|^{1/4} \Delta_g |g|^{-1/4}.$$

Under a conformal rescaling $g = e^{2\sigma}\bar{g}$, the half-density conjugation produces a quadratic-gradient term $\propto |\nabla\sigma|^2$ with universal coefficient $D(4-D)/4$, which cancels at $D = 4$ (within the conformal class). (To see this, note that $|g|^{1/4} = e^{D\sigma/2}|\bar{g}|^{1/4}$; conjugating Δ_g by $e^{D\sigma/2}$ and using the conformal Laplacian identity $\Delta_g \phi = e^{-2\sigma}[\Delta_{\bar{g}}\phi + (D-2)\bar{g}^{ij}\partial_i\sigma\,\partial_j\phi]$ produces the potential $V_{\mathrm{HD}} = -\frac{D}{2}\Delta_{\bar{g}}\sigma + \frac{D(4-D)}{4}|\bar{\nabla}\sigma|^2$; the coefficient $\frac{D(4-D)}{4} = \frac{D^2}{4} - \frac{D(D-2)}{2}$ collects the contributions from the chain rule applied to $e^{-D\sigma/2}$.) This is an operator-simplicity filter (scale-neutral) and is independent of the coupling-dimension argument. A companion satellite on half-densities in QFT develops the full bi-half-density calculus for spacetime propagators and Green functions, including the conformal-class expansion, the normal-coordinate computation showing the universal conjugation potential $V = (1/6)R$ matches conformal coupling only at $D = 4$, and the heat-kernel trace formula without extraneous $\sqrt{|g|}$ factors.

On the renormalization side, $D = 4$ is the unique dimension in which 1-form gauge couplings are marginal by power counting ($[g^2]$ = length^{D-4}), and in which the classical Yang–Mills action is conformally invariant (the action-density weight factor $e^{(D-4)\sigma}$ equals unity). Both this statement and the half-density cancellation share the algebraic root $(D-4) = 0$, but they apply to different objects: the half-density version is universal (independent of matter content), while the RG version is coupling-specific (tied to the form degree of the gauge connection via $D = 2(p+1)$ for a p-form field; $p = 1$ gives $D = 4$). The consistency of these two logically independent filters at $D = 4$ is a nontrivial structural coincidence.

Derivation 6.1 (Time slicing from repeated composition). Iterating the composition law over a partition $t_i = t_0 < \cdots < t_N = t_f$ gives

$$K(q_f, t_f; q_i, t_i) = \int \prod_{k=1}^{N-1} dq_k \prod_{k=0}^{N-1} K_\Delta(q_{k+1}, q_k; t_k),$$

with $q_0 = q_i,\ q_N = q_f,\ \Delta t_k = t_{k+1} - t_k$, and K_Δ the short-time kernel.

Derivation 6.2 (Semigroup fixes the $t^{-d/2}$ normalization). On $M = \mathbb{R}^d$, translation invariance suggests a bi-half-density kernel of the form $K_t(x,y) = k_t(x-y)\,|dx|^{1/2}|dy|^{1/2}$, so the semigroup law becomes a scalar convolution: $k_{t+s} = k_t * k_s$. Assume a quadratic short-time phase and write

$$k_t(u) = A(t)\,\exp\!\left(\frac{im}{2\hbar}\frac{\|u\|^2}{t}\right),$$

interpreting the Gaussian integral in Euclidean time (heat kernel) and then analytically continuing, or with the usual $i0$ prescription. Then

$$(k_t * k_s)(u) = A(t)A(s)\int_{\mathbb{R}^d} \exp\!\left(\frac{im}{2\hbar}\left(\frac{\|u-v\|^2}{t} + \frac{\|v\|^2}{s}\right)\right) dv.$$

Completing the square yields

$$\frac{\|u-v\|^2}{t}+\frac{\|v\|^2}{s}=\frac{\|u\|^2}{t+s}+\frac{t+s}{ts}\left\|v-\frac{s}{t+s}u\right\|^2,$$

so the convolution closes on the same family with

$$A(t+s)=A(t)A(s)\left(\frac{ts}{t+s}\right)^{d/2}\times(\text{phase}).$$

The unique solution (up to an overall constant phase) is $A(t)\propto t^{-d/2}$. The exponent $d/2$ is forced by semigroup composition: it is the half-density "square-root Jacobian" needed for refinement-stable kernel composition. Imposing the delta initial condition as $t\to 0^+$ fixes the remaining normalization constant and forces $\hbar$ into the prefactor (in standard flat-space scalar conventions, $A(t)=(m/2\pi i\hbar t)^{d/2}$ up to phase).

Proposition 6.1 (Necessity of an action-dimensional scale for composition-compatible refinement). Let $M=\mathbb{R}^d$ with Lagrangian $\mathcal{L}(q,\dot{q},t)$ and action $S[q]=\int\mathcal{L}\,dt$ of dimension $[S]=\text{mass}\cdot\text{length}^2\cdot\text{time}^{-1}$. Assume a family of transition kernels $\{K(q_f,t_f;q_i,t_i)\}_{t_f>t_i}$ satisfying:

(C) Composition (semigroup): $K(q_f,t_f;q_i,t_i)=\int_M dq\,K(q_f,t_f;q,t)K(q,t;q_i,t_i)$ for all $t_i<t<t_f$.

(L) Local exponential weight: $K_{\Delta t}(q_f,q_i)=\mathcal{N}(\Delta t)\exp(c_0S_{\text{slice}})$ for short times.

(I) Identity: $K(q_f,t_f;q_i,t_i)\to\delta^{(d)}(q_f-q_i)$ as $t_f\to t_i^+$.

(D) Dimensional homogeneity: $[K]=\text{length}^{-d}$.

Then:

(i) The normalization exponent is $\mathcal{N}(\Delta t)\propto(\Delta t)^{-d/2}$, uniquely determined by composition closure (Derivation 6.2).

(ii) There exists a constant κ with $[\kappa]=[\text{action}]$ such that $\mathcal{N}(\Delta t)=(m/2\pi\kappa\Delta t)^{d/2}$ and $c_0=i/\kappa$ (or $-1/\kappa$ in Euclidean signature), fixed by the identity limit and dimensional analysis.

(iii) κ cannot be eliminated: setting $\kappa\to 0$ collapses composition to Hamilton–Jacobi extremization, which has no distributional identity limit (the composed kernel approaches $\delta(q_f-q_{\text{cl}}(q_i))$, generically not $\delta(q_f-q_i)$ for $t>0$); setting $\kappa\to\infty$ makes the weight trivial and loses dynamical content. The scale κ is therefore a necessary structural constant of any composition-compatible refinement of action-based dynamics.

(iv) The structural result is that κ exists and is necessary; identifying $\kappa=\hbar$ (a physical identification with the empirically measured quantum of action, not a further structural forcing) recovers the standard quantum propagator with Feynman phase $e^{iS/\hbar}$ and Van Vleck normalization.

This is the central result of the paper. The proposition says that any attempt to build a composition-compatible refinement of action-based dynamics is forced to introduce a new dimensional constant with the dimensions of action. Classical mechanics, which lacks such a constant, is structurally incomplete as a refinement theory — not wrong, but unable to support composition closure without an additional scale.

Remark (Literature precedents for composition-forced measure). The idea that composition (the semigroup law) constrains the path-integral measure has three principal antecedents. DeWitt

[DeWitt1957] first identified the Van Vleck–Morette determinant $\Delta^{1/2}(x,y)$ as the required measure factor for path integrals on curved spaces; the factor is a bi-half-density (it transforms as $|g(x)|^{1/4}|g(y)|^{1/4}$ under coordinate changes). Kleinert and Chervyakov [KleinertChervyakov2000] derived $\Delta^{1/2}$ from the global semigroup property alone, showing that composition forces the normalization without appeal to operator-ordering arguments — the closest antecedent to the present derivation. Baldazzi, Percacci, and Zanusso [BaldazziPercacciZanusso2021] showed that "cutting and gluing" (composition across temporal boundaries) determines the normalization of self-normalizing path integrals uniquely. Proposition 6.1 extends these measure-level results to a necessity theorem for the scale $\kappa = \hbar$ itself, operating at the level of the kernel functional equation and reducing the hypothesis set to composition plus dimensional setup.

Remark (Relation to Feynman-Kac semigroup theory). The semigroup property central to Proposition 6.1 is the defining structural axiom of the Feynman-Kac formula [Kac1949FeynmanKac]: for a potential V, the operator $e^{-t(H_0+V)}$ acts on L^2 via the kernel $K(x,y;t) = \mathbb{E}_x[\exp(-\int_0^t V(B_s)\,ds)\,\delta(B_t - y)]$, where B is Brownian motion. Nelson [Nelson1964FeynmanSchrodinger] established the rigorous connection between this probabilistic semigroup and the Schrodinger equation, proving that the Trotter product formula $e^{-t(H_0+V)} = \lim_{n\to\infty}(e^{-tH_0/n}e^{-tV/n})^n$ converges strongly. The present paper does not claim to supersede this framework; rather, it asks a different question. Feynman-Kac theory *assumes* the existence of the semigroup generator H and derives the kernel; Proposition 6.1 works in the reverse direction, deriving the necessity of $\kappa = \hbar$ from the requirement that a kernel satisfying composition exists at all. The two perspectives are complementary: Feynman-Kac provides existence and regularity; composition-forcing provides necessity and uniqueness of the dimensional scale. The time-slicing construction in Derivation 6.1 is the discrete analog of the Trotter product formula, and the regulated-kernel construction in Section 10.6 parallels the heat-semigroup regularization of [Schulman1981PathIntegration, Ch. 6].

Remark (Connection to the Refinement Compatibility Principle). The proposition is the primary constructive witness of the Refinement Compatibility Principle (Section 9): partition compatibility demands a parameter update for composition to close. The fact that this parameter has dimension of action is forced by the dimensional content of the classical Lagrangian, making the principle constructive — it determines what the control parameter must be.

Proposition 6.2 (Exponential weight is forced by composition). Hypothesis **(L)** (local exponential weight) decomposes into two parts: **($\mathbf{L}_{\text{loc}}$)** kernel dependence on classical action only, and **($\mathbf{L}_{\exp}$)** exponential form $W = \exp(c_0 S/\kappa)$. The exponential form is not an independent assumption: for a translation-invariant free kernel $K_t(u) = N(t)W(mu^2/(2\kappa t))$, composition **(C)** in Fourier space reduces to the multiplicative equation $\widehat{K}_T(p) = \widehat{K}_{t_1}(p)\widehat{K}_{t_2}(p)$. Under continuity, the unique solution is $\log \widehat{K}_t(p) = t\,\phi(p)$, which forces $\log \widehat{W}_0(q) = \alpha q^2 + \beta$ (quadratic), i.e., $W = \exp(\text{quadratic})$. The functional equation admits no continuous non-Gaussian solution. For translation-invariant free kernels, the exponential form is a theorem of composition, not an axiom. (For general potentials or curved configuration spaces, the exponential form remains an input.)

Proposition 6.3 (Levy-Khintchine obstruction closes the loophole). The remaining escape from Gaussian kernels is Levy-stable processes with characteristic exponent $\Psi(p) = -c|p|^\alpha$, $\alpha \in (0,2]$. For these, dimensional homogeneity **(D)** requires $[c] = [m]^a[\hbar]^b$. Matching length, mass, and time exponents yields the consistency condition $-\alpha/2 = 1-\alpha$, which forces $\alpha = 2$ (Gaussian). For $\alpha \neq 2$, no combination of m and $\hbar$ can build the required coefficient c. While all Levy processes satisfy the identity limit **(I)**, the discriminating hypothesis is **(D)**: any kernel satisfying **(C)**+**(I)**+**(D)** with

dimensional constants $\{m, \hbar\}$ must be Gaussian. The effective hypothesis count for Proposition 6.1 is therefore reduced from four to three: composition **(C)**, identity limit **(I)**, and dimensional homogeneity **(D)**.

Remark (Non-circular derivation of the action-dimensional constant). A natural objection to Proposition 6.3 is an apparent circularity: the dimensional basis $\{m, \hbar\}$ is used to exclude non-Gaussian processes, but $\hbar$ is the constant being derived. The circularity is resolved by an older argument [Rivero1998Feynman] that makes no dimensional assumption. The exponential weighting of the path integral arises already at the classical level, from the integral (Fourier) representation of the Dirac measure concentrated on critical points of a functional:

$$\langle \delta_L^{\varepsilon,\varepsilon'} | O[\phi] \rangle = \int \cdots \int \frac{1}{\varepsilon^{n/2}} e^{i\frac{1}{\varepsilon}\sum L^{\varepsilon'}[x_i, \frac{x_{i+1}-x_i}{\varepsilon'}]} O[\phi] \prod dx_i,$$

where ε is the distributional regularization parameter (from the Fourier representation of $\delta(f')$) and $\varepsilon' = (t_1 - t_0)/(n+1)$ is the time-slicing scale. Both must go to zero. Convergence of the double limit requires their ratio to be held fixed: $\varepsilon = h\varepsilon'$. The dimensions of h follow from the structure: the exponent $\frac{1}{\varepsilon}\sum L^{\varepsilon'}\varepsilon'$ must be dimensionless; since L has dimensions of energy and ε' has dimensions of time, $L\varepsilon'$ has dimensions of action; therefore $[\varepsilon] = [\text{action}]$ and $[h] = [\varepsilon/\varepsilon'] = [\text{action}]/[\text{time}] \cdot [\text{time}] = [\text{action}]$. No dimensional basis is assumed — the constant emerges from the convergence condition alone.

In the tangent groupoid framework [Rivero1997Grupoid], the parameter ε acquires a geometric identity. The tangent groupoid G is the union of the tangent bundle TM (classical mechanics, $\varepsilon = 0$) and the secant groupoid $SM = \{[x, y, \varepsilon] : x, y \in M, \varepsilon > 0\}$ (finite differences, quantum mechanics), with TM as the boundary of SM. The finite-difference distributions $\langle [x, y, \varepsilon] | f \rangle = (f(x) - f(y))/\varepsilon$ compose via the groupoid law and reduce to directional derivatives $\langle [x, X] | f \rangle = \partial_X f|_x$ at $\varepsilon = 0$. A dilatation structure $\tau_\lambda : SM \to SM$, $[x, y, \varepsilon] \mapsto [x', y', \lambda\varepsilon]$, satisfying the semigroup law $\tau_{\lambda\mu} = \tau_\lambda \circ \tau_\mu$, controls convergence to the boundary: a sequence $\{[x_n, y_n, \varepsilon_n]\}$ with $\varepsilon_n \to 0$ converges to a tangent vector $[x, X] \in TM$ if and only if the rescaled sequence $\{\tau_{\varepsilon_0/\varepsilon_n}[x_n, y_n, \varepsilon_n]\}$ has a limit in SM for some fixed physical scale $\varepsilon_0 > 0$.

This is precisely a renormalization group flow in Wilson-Kogut form [Rivero1997Grupoid, Appendix]. The fixed points of the dilatation flow are $\{[x, x, 0] : x \in M\}$, where distributions become indefinite. A sequence approaching the fixed-point surface traces a "bare" series; the rescaled sequence is the "renormalized" series, living on the surface $\varepsilon = \varepsilon_0$. Renormalized limits at different scales are connected by RG transformations (the relevant direction of the flow). The physical scale ε_0 has dimensions of action — forced by the groupoid structure, not assumed — and plays the role of $\hbar$ in the quantum theory. The entire Wilson-Kogut triangle (bare theory, renormalized theory, continuum limit) is thus encoded in the geometry of the tangent groupoid boundary.

Remark (Extensions: curved, interacting, and Lorentzian settings). The conclusions of Proposition 6.1 are not limited to free particles in flat Euclidean space. Three independent extensions confirm robustness:

(1) **Curved configuration spaces.** On a Riemannian manifold (M, g), the short-time kernel involves the Van Vleck–Morette determinant $\Delta^{1/2}(x, y)$, which composes via the DeWitt–Morette semigroup law. The $d/2$ normalization exponent is forced by the same functional equation; Δ and $\sqrt{|g|}$ are geometric (metric-dependent) but κ-independent. The metric provides no escape route from κ-necessity.

(2) **Interacting systems.** The Mehler kernel for the harmonic oscillator $V = \frac{1}{2}m\omega^2 q^2$ satisfies composition with $\sin \omega t$ replacing t as "time function." The $d/2$ forcing is purely algebraic (dimension-counting), and the Seeley–DeWitt coefficients a_n are composition-compatibility conditions, not escape routes. Short-time extraction of $a_1 = -iV/\hbar$ confirms this.

(3) **Lorentzian signature.** Replacing Gaussian damping by oscillatory Fresnel phase, the algebraic parts of the argument ($d/2$ exponent, $\kappa = \hbar$ necessity, semigroup closure) are signature-independent. The $i\varepsilon$ prescription is itself a theorem of composition: among regulators $\exp(-\varepsilon f((x-y)^2/t))$, only $f(x) = x$ (linear) preserves semigroup closure, and this uniqueness follows from degree-counting (non-linear f produces non-Gaussian integrands). Composition alone — without invoking the identity limit — forces the $i\varepsilon$ form.

Heuristic 6.4 (Universality of κ across interacting sectors). Proposition 6.1 forces a scale κ with $[\kappa] = [\text{action}]$ for each sector individually. If two sectors (say particles A and B) interact, their joint kernel must compose as a single semigroup. Under the single-exponential-weight assumption, the joint kernel inherits a single κ: factorization into sector-specific $\kappa_A \neq \kappa_B$ is incompatible with joint composition when A and B are coupled. Decoupled (non-interacting) sectors may in principle carry independent scales, but transitivity of interaction in any connected physics makes $\kappa = \hbar$ universal across all interacting degrees of freedom. This universality is not postulated but forced by composition.

Remark (Continuum time is forced by dimensional homogeneity). Dimensional homogeneity **(D)** not only forbids jump processes (Proposition 6.3) but also forces continuous time parametrization. A discrete-time formulation with lattice spacing ε would require $[\varepsilon] = T$, but in non-relativistic quantum mechanics the dimensional basis is $\{m, \hbar\}$. Seeking $\varepsilon = [m]^a[\hbar]^b$ yields $M^{a+b}L^{2b}T^{-b} = T$; matching exponents gives $2b = 0$ (length) and $-b = 1$ (time), hence $b = 0$ and $b = -1$ simultaneously — a contradiction. The root cause is that $\{m, \hbar\}$ span only a two-dimensional subspace of the three-dimensional MLT dimension space, and the reachable dimensions satisfy $[L] = [T^{-1/2}]$ (up to mass factors). Thus **(D)** forbids external time scales, forcing the continuum limit $\varepsilon \to 0$. Proposition 6.1 therefore establishes not only the necessity of a finite action scale $\kappa = \hbar$, but also the necessity of continuous time as the composition parameter. (Lattice formulations in QFT introduce ε as an external UV regulator; their validity depends on controlled $\varepsilon \to 0$ limits, consistent with this obstruction.)

Proposition 6.5 (Composition as single master axiom, free flat case). For the translation-invariant free case on $\mathbb{R}^d$, the axiom reduction of the preceding results can be pushed further. Hypothesis **(D)** (dimensional homogeneity) is itself a theorem of **(C)** (composition) on $\mathbb{R}^d$: the composition integral $\int K(x,z;t_1)\,K(z,y;t_2)\,d^dz = K(x,y;t_1+t_2)$ forces $[K] = L^{-d}$ by dimensional consistency, and the remaining content of **(D)** is either trivially true for Lagrangian theories or a property of the physical setup. Similarly, hypothesis **(I)** (identity limit) is derivable: the explicit Gaussian kernel forced by **(C)** is a nascent delta family, so $K(x,y;t) \to \delta(x-y)$ as $t \to 0$ is a theorem. For general potentials, Stone's theorem yields the same conclusion given a self-adjoint generator.

The effective hypothesis count for Proposition 6.1 is therefore one mathematical axiom (composition) plus the physical setup (action-based dynamics on $\mathbb{R}^d$ with mass m and self-adjoint Hamiltonian). Composition is the single master axiom for the partition channel. The clean hypothesis reduction via the Cauchy functional equation (Proposition 6.2) and Levy-Khintchine exclusion (Proposition 6.3) is established rigorously for the translation-invariant free case on $\mathbb{R}^d$; the extensions to curved, interacting, and Lorentzian settings proceed by a different route — semiclassical/Van Vleck/operator-theoretic — that invokes stronger hypotheses than composition alone.

The "single master axiom" label applies without qualification only to the flat free case; the extended cases share the same conclusion but via additional input. See [RCPFoundations] Section 6 for the full axiom-reduced formulation.

Remark (Parallel reconstruction: information-theoretic route). Luiz and de Oliveira [LuizOliveira2026] independently arrive at overlapping conclusions via a different starting point: a density of action states $g(A; b, T|a)$ satisfying a composition law in action space, finite variance, and locality. Their Proposition 1 derives the Gaussian form via the same Levy-Khintchine exclusion, and their $d/2$ exponent matches ours. Their route additionally derives complex amplitudes from Cramer-Rao indistinguishability — a step we assume (the exponential weight is here forced by the Cauchy functional equation, Section 6.2). Conversely, our approach yields the groupoid reading (Section 7.4), the RCP three-channel framework (Section 9), and the deformation-quantization bridge [TangentGroupoidBridge], none of which appear in [LuizOliveira2026]. The shared core — convolution semigroup structure forces an action-dimensional scale — is standard mathematics (the Levy-Khintchine theorem applied to propagator kernels); the convergence of two independent lines of argument strengthens the case that composition is the operative principle.

A complementary *kinematic* derivation by Goyal, Knuth, and Skilling [GoyalKnuthSkilling2010] shows that composition of sequential measurement amplitudes, together with continuity, forces complex arithmetic and the Born rule — recovering Feynman's sum and product rules without postulating complex amplitudes. Where [LuizOliveira2026] and the present work derive *dynamical* structure ($\hbar$, Gaussian kernels), [GoyalKnuthSkilling2010] derives *algebraic* structure (the complex field); the three results are complementary instances of the composition-forcing programme. The GKS framework has been extended to a coordinate-independent classification of allowable amplitude algebras (complex and quaternionic) via Hurwitz's composition-algebra theorem, deriving the Born rule without postulating the complex field [Koplinger2025]. The broader programme of deriving quantum theory from operational/compositional axioms includes Hardy's five-axiom reconstruction [Hardy2001FiveAxioms] and the information-theoretic derivation of Chiribella, D'Ariano, and Perinotti [Chiribella2011InformationalDerivation]; these work at the level of the state-space structure (density matrices, purification, tomography) rather than the propagator kernel, and their composition axiom is compositional closure of parallel and sequential operations rather than temporal semigroup composition. The present work and [LuizOliveira2026] are closer to the semigroup/propagator tradition; the operational reconstructions are closer to the quantum-logic/convex-state tradition.

6.2 From Additive Action to Multiplicative Weights

The Section 4/Section 5 structure gives an additive discrete action:

$$S_N[q] = \sum_{k=0}^{N-1} \mathcal{L}\left(q_k, \frac{q_{k+1} - q_k}{\Delta t_k}, t_k\right) \Delta t_k.$$

Assume short-time locality: each slice contributes a factor depending only on local step data. Write

$$K_\Delta(q_{k+1}, q_k; t_k) = \mathcal{N}_k \, W_k.$$

Proposition 6.6 (Exponential form under locality + composition). If

1. total path weight is multiplicative across concatenated slices, and

2. $\log W_k$ is additive in Δt_k to first order (equivalently, $W_k = \exp(c_0 \mathcal{L}_k \Delta t_k + O(\Delta t^2))$; the derivation that composition *forces* this form via the Cauchy functional equation is in Proposition 6.2),

then, up to normalization and higher-order slicing corrections,

$$\prod_{k=0}^{N-1} W_k \propto \exp(c_0\, S_N[q]),$$

for a constant c_0 with dimensions $[\text{action}]^{-1}$.

Choosing oscillatory quantum time evolution gives $c_0 = i/\hbar$ (the necessity of this scale — dimensional homogeneity forces $[c_0] = [\text{action}]^{-1}$ and the identity limit pins the coefficient — is derived in Proposition 6.1(ii)), hence the standard phase factor $\exp(iS_N/\hbar)$ [Dirac1933] [Feynman1948].

6.3 Ordering, Discretization, and Quantum Ambiguity

Different short-time discretizations (left/right/midpoint or more general α-schemes) typically correspond to different operator orderings. In deformation language, this is the same ambiguity as choosing a star-product representative; these constructions agree in the classical limit but can differ at subleading quantum order [Landsman1998] [deGosson2018ShortTimePropagators].

The key point is not that this ambiguity is a problem, but that it is *controlled*: two discretizations that differ by $O(\Delta t)$ in each slice produce equivalent classical equations while shifting $O(\hbar)$ terms in quantum generators. Ordering is a modeling choice, not a contradiction.

Derivation 6.3 (Ordering witness: $H = qp$ under α-discretization). For the classical Hamiltonian $H(q,p) = qp$, the phase-space path integral with α-prescription (evaluating position at $q_\alpha = (1-\alpha)q_k + \alpha q_{k+1}$ in each slice) produces the operator

$$\hat{H}_\alpha = \alpha \hat{q}\hat{p} + (1-\alpha)\hat{p}\hat{q} = \hat{p}\hat{q} + \alpha\, i\hbar.$$

To verify: the matrix element $\langle q'|\hat{H}_\alpha|q\rangle = [\alpha q' + (1-\alpha)q](-i\hbar)\delta'(q'-q) = q_\alpha(-i\hbar)\delta'(q'-q)$, which matches the kernel's α-interpolated integrand order by order in Δt. At $\alpha = 0$ (prepoint) this is anti-standard ordering $\hat{p}\hat{q}$; at $\alpha = 1/2$ (midpoint) it is the Weyl-symmetric form $\frac{1}{2}(\hat{q}\hat{p} + \hat{p}\hat{q})$ connected to the Moyal product of Section 7; at $\alpha = 1$ (postpoint) it is standard ordering $\hat{q}\hat{p} = \hat{p}\hat{q} + i\hbar$. All three share the classical limit $H = qp$; the $O(\hbar)$ corrections are discretization artifacts controlled by α. This is the simplest instance of the general $f(q)p$ family treated in Section 10.2.

Remark (Symmetry selects the midpoint). In the position representation $\hat{H}_\alpha = -i\hbar(q\frac{d}{dq} + (1-\alpha))$ on $L^2(\mathbb{R}, dq)$, the formal symmetry defect is $\langle u, \hat{H}_\alpha v\rangle - \langle \hat{H}_\alpha u, v\rangle = i\hbar(2\alpha - 1)\langle u, v\rangle$ for test functions with compact support away from $q = 0$. This vanishes if and only if $\alpha = 1/2$. The logical chain is: unitarity of the generated group requires self-adjointness of the generator, self-adjointness requires symmetry, and symmetry selects $\alpha = 1/2$. Essential self-adjointness (closure to a unique self-adjoint extension) additionally depends on the configuration space: it holds automatically on $\mathbb{R}_+$ via the dilation group but requires choosing a self-adjoint extension on $\mathbb{R}$ where the singularity at $q = 0$ creates deficiency indices $(1, 1)$. The $\alpha = 1/2$ selection itself is a purely algebraic consequence of the L^2 inner product and is independent of these domain subtleties.

Remark (Discretization-stochastic dictionary: Ito vs Stratonovich). The α-discretization maps directly to stochastic calculus conventions. For an N-impulse model with position-dependent random

impulses $J_k = \sigma(q_k^*)\sqrt{\Delta t}\,\xi_k$ (where ξ_k are i.i.d. standard normals and q_k^* is the evaluation point), the three standard discretizations correspond to: $\alpha = 0$ (prepoint $q_k^* = q_k$) yields the Ito integral; $\alpha = 1/2$ (midpoint $q_k^* = (q_k + q_{k+1})/2$) yields the Stratonovich integral; $\alpha = 1$ (postpoint $q_k^* = q_{k+1}$) yields the anti-Ito or backward integral. The Stratonovich-to-Ito conversion formula adds a correction term $\frac{1}{2}\sigma(q)\sigma'(q)\,dt$ to the drift, which has the same structural origin as the α-dependent $O(\hbar)$ shift in the quantum Hamiltonian: both are "chain-rule corrections" arising from moving the evaluation point in a context where increments have $O(\sqrt{\Delta t})$ fluctuations. The Stratonovich convention preserves the classical chain rule and the variational impulse-work formula; the Ito convention has better martingale properties but breaks manifest symmetry.

Remark (Composition selects half-density, not midpoint, on curved manifolds). On flat configuration space $\mathbb{R}^d$, the midpoint discretization ($\alpha = 1/2$, Weyl ordering, Stratonovich convention) and the half-density conjugation $|g|^{1/4}(-\Delta_g)|g|^{-1/4}$ produce operators that agree on the principal symbol and connection terms, differing only in an $O(\hbar^2)$ scalar potential (Section 9). On curved Riemannian manifolds, however, these two selections diverge. The midpoint/Stratonovich prescription produces a scalar curvature coupling $\xi_W = 1/8$ (i.e. $V = R/8$), while the half-density conjugation produces $V_{\mathrm{HD}} = -R/6$ at each Riemann normal coordinate center (Section 9). The composition law takes precedence: the Van Vleck–Morette determinant that appears in the semiclassical kernel (Derivation 6.5 below) transforms as a bi-half-density, and kernel composition requires the $|g|^{1/4}$ prefactor, not the midpoint evaluation point. (This step uses the additional assumption that the kinetic operator is defined by the half-density conjugation $\Delta_{1/2} = |g|^{1/4}(-\Delta_g)|g|^{-1/4}$; composition determines the measure coefficient universally via Proposition 6.1, and the conjugation then specifies the corresponding operator ordering. On general manifolds the half-density $|g|^{1/4}$ has no unique algebraic origin beyond Proposition 6.1 itself, in contrast to the Lie group case where it coincides with the Duflo factor, cf. Section 9.)

The hierarchy of selection principles on curved manifolds is therefore: composition (half-density, $R/6$) $\supset$ symmetry (midpoint, $R/8$) $\supset$ classical limit (any α). The first two coincide on flat space where all curvature couplings vanish.

6.4 Formal Continuum Limit and Stationary Phase

Formally, as mesh size $\max_k \Delta t_k \to 0$:

$$K(q_f, t_f; q_i, t_i) \sim \int_{q_i}^{q_f} \mathcal{D}q\ \exp\!\left(\frac{i}{\hbar} S[q]\right).$$

This expression is formal: there is no countably additive measure on full path space (cf. Section 1). For an explicit regulated-kernel family with exact composition and controlled regulator removal in the Euclidean free and harmonic-oscillator models, see Appendix 10.6.

Derivation 6.4 (Classical recovery via stationary phase). Let $q = q_{\mathrm{cl}} + \xi$, where q_{cl} is stationary: $\delta S[q_{\mathrm{cl}}; \eta] = 0$. Expand:

$$S[q_{\mathrm{cl}} + \xi] = S[q_{\mathrm{cl}}] + \frac{1}{2}\langle \xi, \mathcal{H}_{q_{\mathrm{cl}}} \xi \rangle + O(\xi^3).$$

Fast phase oscillations cancel nonstationary contributions, while neighborhoods of stationary paths contribute coherently. This is the precise sense in which the classical equations reappear as $\hbar \to 0$ asymptotics.

Derivation 6.4a (Soft extremum and why a finite $\hbar$ stabilizes refinement). Even before any infinite-dimensional measure issues, the refinement-composition pattern forces a "softened" version of extremization. Consider a finite-dimensional two-slice action $S(q_2, q_1) + S(q_1, q_0)$ and form the Euclideanized composed weight

$$W_\hbar(q_2, q_0) := \int dq_1 \ \exp\Big(-\frac{1}{\hbar}(S(q_2, q_1) + S(q_1, q_0))\Big).$$

Define the associated coarse effective action (a log-partition functional)

$$S^{(\hbar)}_{\text{eff}}(q_2, q_0) := -\hbar \ln W_\hbar(q_2, q_0).$$

Then refinement-composition is exact at the level of weights (add actions in the exponent, integrate the intermediate variable), and the hard elimination/extremization rule appears only as the sharpening limit: under standard nondegeneracy assumptions, Laplace's method gives

$$S^{(\hbar)}_{\text{eff}}(q_2, q_0) \xrightarrow{\hbar\to 0} \inf_{q_1}(S(q_2, q_1) + S(q_1, q_0)),$$

with $O(\hbar)$ corrections determined by local quadratic data near the minimizer(s). So $\hbar$ plays the role of a universal control parameter that makes "refine then compare" stable, with classical extremals recovered as a limit. In real time, the same pattern appears with $e^{iS/\hbar}$ and stationary phase in place of positivity and Laplace concentration.

Example (Free-particle two-slice integral: explicit classical recovery). For a free particle of mass m, the two-slice action is $S = \frac{m}{2\Delta t}[(q_2 - q_1)^2 + (q_1 - q_0)^2]$. Completing the square gives $S = \frac{m}{4\Delta t}(q_2 - q_0)^2 + \frac{m}{\Delta t}(q_1 - \bar{q})^2$ with $\bar{q} = (q_0 + q_2)/2$. The Euclideanized weight integral is Gaussian:

$$W_\hbar(q_2, q_0) = \sqrt{\frac{\pi\hbar\Delta t}{m}} \ \exp\left(-\frac{m(q_2 - q_0)^2}{4\hbar\Delta t}\right),$$

giving the coarse effective action

$$S^{(\hbar)}_{\text{eff}}(q_2, q_0) = \frac{m(q_2 - q_0)^2}{4\Delta t} + \frac{\hbar}{2}\ln\frac{m}{\pi\hbar\Delta t}.$$

As $\hbar \to 0$, $S^{(\hbar)}_{\text{eff}} \to \inf_{q_1} S = \frac{m(q_2-q_0)^2}{4\Delta t}$ with a subleading $O(\hbar \ln \hbar)$ correction from the Gaussian width. The extremal path ($q_1 = \bar{q}$, uniform velocity) is selected by sharpening, with no additional hypotheses.

Remark (Delocalized angles in angular-momentum eigenstates). The stationary-phase mechanism explains how classical trajectories reappear in semiclassical packets. It does not imply that a single stationary eigenstate is a localized classical orbit. A simple witness occurs in central potentials: in polar coordinates the azimuthal angle ϕ is conjugate to L_z, and separation of variables yields [TongQMLectures]

$$\psi(r, \phi) = R(r)\, e^{im_\ell \phi}.$$

Therefore

$$|\psi(r, \phi)|^2 = |R(r)|^2$$

is independent of ϕ, i.e., the azimuthal angle is uniformly distributed in an L_z eigenstate. Localized "orbit phase" pictures correspond to coherent superpositions/wavepackets, consistent with the manuscript's use of stationary-phase concentration rather than "eigenstate = orbit" identification.

(A companion note on action-angle indeterminacy expands this observation to a second witness — the harmonic oscillator, where Fock states delocalize the orbit phase while coherent states localize it, with decoherence dynamically selecting the localized packets — and formulates the general structure as an uncertainty relation between action and angle variables.)

Derivation 6.5 (Van Vleck prefactor is a bi-half-density). In the semiclassical approximation, the endpoint kernel has the standard form

$$K(q_f, t_f; q_i, t_i) \approx \frac{1}{(2\pi i\hbar)^{d/2}} \left|\det\left(-\frac{\partial^2 S_{\mathrm{cl}}}{\partial q_f\, \partial q_i}\right)\right|^{1/2} \exp\left(\frac{i}{\hbar} S_{\mathrm{cl}}(q_f, t_f; q_i, t_i)\right),$$

where S_{cl} is the classical action evaluated on the stationary path with those endpoints. Under changes of coordinates $q_f = q_f(q_f')$, $q_i = q_i(q_i')$, the mixed Hessian transforms by the chain rule, and

$$\det\left(-\frac{\partial^2 S_{\mathrm{cl}}}{\partial q_f'\, \partial q_i'}\right) = \det\left(\frac{\partial q_f}{\partial q_f'}\right) \det\left(\frac{\partial q_i}{\partial q_i'}\right) \det\left(-\frac{\partial^2 S_{\mathrm{cl}}}{\partial q_f\, \partial q_i}\right).$$

Taking square roots yields a factor $|\det(\partial q_f/\partial q_f')|^{1/2} |\det(\partial q_i/\partial q_i')|^{1/2}$, so the prefactor transforms as a half-density in each endpoint variable. This is the concrete semiclassical origin of the half-density viewpoint introduced at the start of this section. For an early semiclassical/correspondence-principle anchor in the "Van Vleck" tradition, see [VanVleck1928Correspondence]. For a modern OA statement of the Van Vleck propagator/prefactor and the associated "Van Vleck density", see [deGosson2018ShortTimePropagators].

Remark (Caustics and the Maslov index). At caustics (conjugate points where the Van Vleck determinant $D \to \infty$), the semiclassical approximation appears to break down: the amplitude $|D|^{1/2}$ diverges and $\sqrt{D}$ changes sign. In the half-density framework the singularity is a projection artifact: the semiclassical state is a smooth half-density on the Lagrangian submanifold $L \subset T^*M$, and the caustic occurs because the projection $\pi : L \to M$ develops a fold [BatesWeinstein1997]. The sign ambiguity of $\sqrt{D}$ is resolved by the metaplectic structure: half-densities transform under the double cover $\mathrm{Mp}(2n)$ of $\mathrm{Sp}(2n)$, and the Maslov index μ — counting conjugate points along the classical path — records the accumulated phase correction $e^{-i\pi\mu/2}$. This makes the "amplitudes are half-densities" pattern precise: the Van Vleck prefactor is not a scalar but a section of a metalinear bundle, globally well-defined even through caustics.

6.5 Link Back to Section 5 Singular Dynamics

The composition picture naturally includes piecewise-smooth trajectories. At impulses, the dominant classical skeleton must satisfy the jump laws from Section 5:

$$\left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\right]_{t_0^-}^{t_0^+} = J \quad \text{(impulse)} \qquad \left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\right]_{t_0^-}^{t_0^+} = 0 \quad \text{(corner, unforced)}.$$

The "extremal set" entering semiclassical evaluation is broader than globally smooth trajectories; it includes admissible broken trajectories obeying the correct matching conditions.

Derivation 6.6 (Impulse-kick kernel: composition produces the jump law). For a free particle on $\mathbb{R}$ subject to a single instantaneous impulse J at time $t_0 \in (0, T)$, the propagator factorizes through the composition integral

$$K_J(x_f, T; x_i, 0) = \int dy\, K_{\text{free}}(x_f, T; y, t_0)\, e^{iJy/\hbar}\, K_{\text{free}}(y, t_0; x_i, 0),$$

where the phase $e^{iJy/\hbar}$ encodes the impulse action $J \cdot q(t_0)$. Completing the Gaussian integral gives:

$$K_J = K_{\text{free}}(x_f, T; x_i, 0)\ \exp\Big[\frac{iJ\,\bar{x}(t_0)}{\hbar} - \frac{iJ^2\, t_0(T-t_0)}{2m\hbar T}\Big],$$

with $\bar{x}(t_0) = x_i + (x_f - x_i)t_0/T$ the unforced classical midpoint. The saddle point $y^* = \bar{x}(t_0) - Jt_0(T-t_0)/(mT)$ satisfies $\Delta p = m(x_f - y^*)/(T-t_0) - m(y^* - x_i)/t_0 = J$, recovering the Weierstrass-Erdmann impulse condition of Section 5 as a *derived consequence* of stationarity, not an additional input. The limits $J \to 0$ (free particle) and $J \to \infty$ (decoupling) are both physically correct. # 7. Deformation Quantization Bridge

7.1 From Path Weights to Product Deformation

Discretized composition introduces nonunique short-time prescriptions (left/right/midpoint and related schemes). The algebraic restatement: quantization deforms the classical product of observables rather than replacing classical mechanics by unrelated objects [Landsman1998] [Connes1994].

Let M be phase space with Poisson bracket $\{\cdot,\cdot\}$, and let $\mathcal{A}_0$ be a commutative algebra of classical observables (e.g., smooth functions with suitable decay/domain conditions). A deformation quantization is a family of associative products $\star_\hbar$ on $\mathcal{A}_0$ such that:

$$f \star_\hbar g = fg + \sum_{n \geq 1} \hbar^n B_n(f, g),$$

where B_n are bilinear operators, with $f \star_0 g = fg$.

Proposition 7.1 (Classical compatibility conditions). If $\star_\hbar$ is associative for each $\hbar$ and depends smoothly/formally on $\hbar$, then the antisymmetric part of B_1 controls the leading noncommutativity:

$$[f, g]_{\star_\hbar} \equiv f \star_\hbar g - g \star_\hbar f = \hbar(B_1(f, g) - B_1(g, f)) + O(\hbar^2).$$

First-order noncommutativity is fully determined by B_1^{anti}.

7.2 Commutator-to-Poisson Recovery

Derivation 7.2 (Correspondence limit). Impose the correspondence requirement that first-order antisymmetry matches the Poisson bracket:

$$B_1(f, g) - B_1(g, f) = i\,\{f, g\}.$$

Then

$$[f, g]_{\star_\hbar} = i\hbar\,\{f, g\} + O(\hbar^2),$$

and therefore

$$\lim_{\hbar\to 0}\frac{1}{i\hbar}[f,g]_{\star_\hbar}=\{f,g\}.$$

Dimensional closure: $[\hbar]=[\text{action}]$, while $\{f,g\}$ carries one inverse action factor relative to fg in canonical coordinates, so $i\hbar\{f,g\}$ has the same physical dimension as fg. This is the same unit-consistency condition already used in Section 6 for $\exp(iS/\hbar)$.

7.3 Concrete Model and Ordering Content

For flat phase space, the Moyal product provides an explicit representative:

$$(f\star_M g)(q,p)=f(q,p)\exp\!\left[\frac{i\hbar}{2}\left(\overleftarrow{\partial_q}\overrightarrow{\partial_p}-\overleftarrow{\partial_p}\overrightarrow{\partial_q}\right)\right]g(q,p),$$

which reproduces the Poisson bracket at leading order and higher quantum corrections at higher orders [Groenewold1946ElementaryQM] [Moyal1949StatisticalQM] [Landsman1998].

Derivation 7.3a (Moyal product for linear and quadratic observables). For the canonical pair $f=q$, $g=p$, the exponential terminates at first order (all higher derivatives vanish):

$$(q\star_M p)(q,p)=qp+\frac{i\hbar}{2}(\partial_q q)(\partial_p p)=qp+\frac{i\hbar}{2}.$$

By symmetry, $(p\star_M q)=pq-\frac{i\hbar}{2}$, so $[q,p]_{\star_M}=i\hbar$, exactly reproducing the canonical commutation relation with no higher-order corrections (as expected for linear functions).

For $f=q^2$, $g=p$, the second-order derivative $\partial_q^2(q^2)=2$ is constant, but it multiplies $\partial_p^2 p=0$, so the expansion again terminates at first order:

$$(q^2\star_M p)=q^2p+\frac{i\hbar}{2}(2q)(1)=q^2p+i\hbar\,q.$$

Thus $[q^2,p]_{\star_M}=2i\hbar\,q$, matching $[\hat{q}^2,\hat{p}]=2i\hbar\,\hat{q}$ in Weyl ordering. More generally, $[q^n,p]_{\star_M}=ni\hbar\,q^{n-1}$, recovering the Leibniz rule for the Poisson bracket $\{q^n,p\}=nq^{n-1}$ at leading order with no further corrections.

Derivation 7.3b (Cubic witness: genuine $O(\hbar^3)$ correction in the Moyal bracket). For pure monomials $f=q^m$, $g=p^n$, all cross-derivatives vanish ($\partial_p q^m=\partial_q p^n=0$), so the Moyal product terminates at order $\min(m,n)$ in $\hbar$:

$$[q^m,p^n]_{\star_M}=\sum_{\substack{k=1\\ k\ \text{odd}}}^{\min(m,n)}\frac{2(i\hbar/2)^k}{k!}\,\frac{m!\,n!}{(m-k)!\,(n-k)!}\,q^{m-k}p^{n-k}.$$

When $\min(m,n)\le 2$ only the $k=1$ term contributes, giving $i\hbar\{q^m,p^n\}$ exactly — the Poisson bracket suffices, as verified in Derivation 7.3a. The first case requiring a higher term is $m=n=3$: the $k=1$ term gives $9i\hbar\,q^2p^2=i\hbar\{q^3,p^3\}$, and the $k=3$ term gives $2(i\hbar/2)^3/3!\cdot(3!)^2=-\frac{3}{2}\,i\hbar^3$, so

$$[q^3,p^3]_{\star_M}=i\hbar\{q^3,p^3\}-\tfrac{3}{2}\,i\hbar^3.$$

The constant $-\frac{3}{2}i\hbar^3$ is a genuine quantum correction that cannot be recovered from the Poisson bracket. This makes the cubic pair the simplest witness that deformation quantization goes beyond a re-encoding of the Poisson algebra.

Remark (Ordering as deformation gauge choice). The Section 6 discretization ambiguity is naturally interpreted as choosing different but deformation-equivalent star products; they share the same classical bracket data but differ in $O(\hbar)$ and higher corrections [Landsman1998].

7.4 Equivalence Classes and Groupoid Viewpoint

Proposition 7.4 (Equivalent star products, same classical limit). If two products $\star_\hbar$ and $\star'_\hbar$ are related by a formal automorphism

$$T_\hbar = \mathrm{id} + \hbar T_1 + O(\hbar^2), \qquad f \star'_\hbar g = T_\hbar^{-1}((T_\hbar f) \star_\hbar (T_\hbar g)),$$

then they define the same Poisson bracket in the $\hbar \to 0$ limit, while generally differing in subleading quantum terms.

Quantization data are organized into equivalence classes compatible with one classical limit. Geometric deformation programs (including tangent-groupoid viewpoints) encode the same bridge from commutative classical data to noncommutative quantum products [Connes1994]. (A companion satellite develops the tangent-groupoid bridge explicitly: the pair groupoid encodes the sewing law, Connes' tangent groupoid interpolates between classical and quantum composition, and the resulting $\hbar$-deformation recovers the path-integral kernel as a groupoid convolution [TangentGroupoidBridge].)

Remark (Deformation equivalence is physical, not merely formal). The formal-automorphism statement of Proposition 7.4 is not purely asymptotic: explicit calculations for a position-dependent mass $f(q) = 1 + \alpha q^2$ in a harmonic trap show that different prescriptions (Weyl, half-density conjugation) agree exactly on the principal symbol (Layer 1) and the first-derivative connection term (Layer 2), differing only in an $O(\hbar^2)$ scalar potential (Layer 3). The resulting ground-state energy shift between prescriptions is $|\Delta E_0| = \hbar\omega\alpha_0^2/16$, where α_0 is dimensionless mass variation at the oscillator scale. For realistic semiconductor heterostructures (GaAs quantum wells, $\hbar\omega \sim 10$ meV, $\alpha_0 \lesssim 0.3$), this gives $|\Delta E_0| \lesssim 0.06$ meV, well below current spectroscopic resolution (~ 0.1 meV). The four-layer stratification and explicit energy estimate are detailed in Appendix 10.2. The physical content of deformation equivalence is that ordering prescriptions are observationally indistinguishable at accessible scales, while the half-density prescription provides a geometrically distinguished representative within the equivalence class.

(A companion satellite on ordering equivalence develops the four-layer classification — principal symbol, connection, scalar potential, domain — systematically, with additional worked examples and connects the ordering/discretization freedom to star-product automorphisms and the Ito/Stratonovich correspondence in path-integral time-slicing.)

7.5 Formal Deformation Boundary

In this section we use formal/asymptotic deformation language for local bridge statements. We do not require the full C^*-algebraic deformation-quantization program for the manuscript's main argument; the needed ingredient is compatibility of the classical limit and quantum corrections under the stated assumptions [Landsman1998].

With the deformation bridge in place, the remaining problem is not how to define first-order quantum corrections, but how to keep refined predictions finite and scale-consistent when naive limits diverge. That control problem is precisely the renormalization step.

8. Renormalization as Controlled Refinement

8.1 Why Renormalization Appears in Refinement Limits

The previous sections treated refinement as benign: polygonal refinement in Section 3, time-slicing in Section 6, and deformation parameter limits in Section 7. In quantum field theory and in several singular quantum-mechanical models (e.g. contact interactions), the same refinement step can instead *diverge* [ManuelTarrach1994PertRenQM] [BoyaRivero1994Contact]: as the cutoff scale is removed, intermediate quantities blow up even when low-energy physics is expected to remain finite.

Renormalization restores the program's central thesis in the divergent case: it provides a controlled rule for taking refinement limits so that observables remain stable. Operationally, it accepts that intermediate expressions depend on a regulator (cutoff), but requires that properly defined observables do not.

Remark (Renormalization as part of "what a theory is"). In benign refinement problems, one can often send the refinement parameter to zero without further choices. In divergent refinement problems, the renormalization prescription — what is regulated, what is held fixed, and how parameters are re-expressed as the cutoff moves — is not optional bookkeeping: it is part of the definition of the continuum theory, because it specifies which composed/refined predictions are declared physically stable.

8.2 Regulator, Bare Data, and Renormalized Observables

Let Λ denote a refinement cutoff (e.g., momentum cutoff $|k| < \Lambda$ or lattice spacing a with $\Lambda \sim 1/a$). Let $g_B(\Lambda)$ denote the cutoff-dependent *bare* parameters of the regulated theory (couplings, masses, field normalizations), and let O_Λ be a regulated prediction for some observable O.

Proposition 8.1 (Renormalized observable as cutoff-stable limit). If there exists a choice of cutoff-dependent bare parameters $g_B(\Lambda)$ such that the limit

$$O_{\text{phys}} \equiv \lim_{\Lambda \to \infty} O_\Lambda(g_B(\Lambda))$$

exists and is finite (or has a controlled asymptotic expansion) for the observables of interest, then the refinement limit is *defined* by this prescription.

This statement is intentionally operational: it does not assume that the cutoff-free object exists without tuning. It states that "physical theory" means a stable target under refinement.

It is often convenient to introduce a renormalization scale μ (a reference resolution) and a renormalization map $R_{\Lambda \to \mu}$ from bare to renormalized parameters:

$$g_R(\mu) = R_{\Lambda \to \mu}(g_B(\Lambda)).$$

The composition viewpoint imposes a compatibility condition: renormalizing from Λ down to μ must be the same as renormalizing from Λ to an intermediate scale κ and then from κ to μ:

$$R_{\Lambda\to\mu} = R_{\kappa\to\mu} \circ R_{\Lambda\to\kappa}.$$

This is the renormalization-group (RG) semigroup property in refinement language; for a standard Wilsonian/ERG discussion of coarse-graining flows and fixed points, see [Rosten2012ERG].

Derivation 8.2 (Control map τ: comparing refinements at fixed ruler). The same compatibility condition can be stated without committing to a particular regulator. Fix a reference ruler $h > 0$ (the resolution at which we compare predictions), and let $A_{h,\theta}$ denote a family of amplitudes/prediction functionals indexed by parameters θ (couplings, normalizations, and any fixed conventions such as scalarization gauge). For a refinement factor $b > 1$, choose a "compare at ruler h" operation $\mathcal{C}_{b,h}$: take a prediction written at finer ruler h/b and express it back at ruler h (e.g. by composing fine steps or integrating intermediate variables when such a representation exists). Scale compatibility is the closure requirement that refinement-comparison lands back in the same family after a parameter update:

$$\mathcal{C}_{b,h}(A_{h/b,\theta}) = A_{h,\tau_b(\theta)}.$$

Here τ_b is the control/flow map induced by the compare-at-fixed-ruler operation. When no such τ_b exists within the chosen parameter family, refinement generates new operators and the family must be enlarged (counterterms/effective operators). A concrete micro-witness is Derivation 8.5a, where step-halving induces $\tau_2(a) = a/2 + 1/4$ with fixed point $a_* = 1/2$. In the cutoff notation above, one may view $R_{\Lambda\to\mu}$ as a special case of τ written with explicit reference scales.

8.3 RG Equation from Cutoff Independence

Derivation 8.3 (RG equation as consistency under refinement). Assume a regulated observable depends on the cutoff Λ both explicitly and through the bare parameters $g_B(\Lambda)$:

$$O_{\text{phys}} = O_\Lambda(g_B(\Lambda)),$$

and impose cutoff-independence of the physical prediction:

$$\frac{d}{d\ln\Lambda} O_\Lambda(g_B(\Lambda)) = 0.$$

By the chain rule,

$$0 = \frac{\partial O_\Lambda}{\partial \ln\Lambda} + \sum_a \frac{dg_B^a}{d\ln\Lambda} \frac{\partial O_\Lambda}{\partial g_B^a},$$

where a ranges over the components of the parameter vector. Defining the beta functions $\beta_B^a(g_B) \equiv \frac{dg_B^a}{d\ln\Lambda}$, we obtain the RG equation:

$$\left(\frac{\partial}{\partial \ln \Lambda} + \sum_a \beta_B^a(g_B) \frac{\partial}{\partial g_B^a} \right) O_\Lambda(g_B) = 0.$$

In the μ-parametrized form with renormalized parameters $g_R(\mu)$, the same reasoning yields a flow equation $\mu \frac{d}{d\mu} g_R(\mu) = \beta(g_R)$ plus corresponding RG invariance equations for renormalized observables. The key structural point: RG is not extra physics added after quantization; it is the *consistency condition* that makes composed refinement meaningful when naive limits diverge.

Proposition 8.2 (Flow generator from refinement semigroup). Let $W_{\Lambda\to\kappa}$ be the map sending effective parameters at scale Λ to effective parameters at scale $\kappa < \Lambda$, and assume: 1. $W_{\Lambda\to\Lambda} = \mathrm{id}$, 2. $W_{\kappa\to\mu} \circ W_{\Lambda\to\kappa} = W_{\Lambda\to\mu}$, 3. differentiability with respect to $\ln \Lambda$.

Then the infinitesimal generator defines beta functions:

$$\beta^a(g) = \frac{d}{dt} W^a_{e^t\mu\to\mu}(g)\Big|_{t=0},$$

and finite scale changes are recovered by integrating this vector field on parameter space. RG flow is the differential form of composed refinement.

Derivation 8.4 (Toy logarithmic divergence and subtraction). Consider a single-coupling situation with a logarithmic cutoff dependence in a regulated prediction:

$$O_\Lambda(g_B; \mu) = g_B + c\, g_B^2 \ln\left(\frac{\Lambda}{\mu}\right) + O(g_B^3),$$

where c is a dimensionless constant determined by the model and by the chosen renormalization convention. Define the renormalized coupling at scale μ by a renormalization condition $g_R(\mu) \equiv O_\Lambda(g_B(\Lambda); \mu)$ which is held fixed as $\Lambda \to \infty$. Cutoff-independence then enforces:

$$0 = \frac{d}{d\ln\Lambda} g_R(\mu) = \frac{dg_B}{d\ln\Lambda} + c\, g_B^2 + O(g_B^3),$$

so $\beta_B(g_B) \equiv \frac{dg_B}{d\ln\Lambda} = -c\, g_B^2 + O(g_B^3)$. Equivalently, at fixed bare coupling one finds the running with μ:

$$\beta(g_R) \equiv \mu \frac{dg_R}{d\mu} = -c\, g_R^2 + O(g_R^3),$$

illustrating how renormalization turns the divergent $\ln\Lambda$ refinement effect into a scale-dependent coupling consistent with stable observables.

8.4 Refinement Control Before QFT: Scale-Halving as a Model

One can see the same logic in purely classical numerical refinement. Consider an evolution operator Φ_ε representing "one step" at resolution ε. Composition gives $\Phi_{2\varepsilon} \approx \Phi_\varepsilon \circ \Phi_\varepsilon$. The refinement-control question: what correction to Φ_ε makes the two-step composition agree with a one-step method after rescaling back to the same reference resolution?

Derivation 8.5a (Step-halving induces a control map τ in a toy ODE). Consider the autonomous ODE $y' = f(y)$ on $\mathbb{R}^n$ and a one-parameter family of one-step maps at step size h,

$$\Phi_h^{(a)}(y) = y + hf(y) + a\,h^2 f'(y)[f(y)] + O(h^3).$$

Here $f'(y)$ is the Jacobian (derivative), and $f'(y)[v]$ denotes its action on an increment v. Define the step-halving comparison $H(\Phi_h) := \Phi_{h/2} \circ \Phi_{h/2}$. A direct expansion to order h^2 gives

$$H(\Phi_h^{(a)})(y) = y + hf(y) + \left(\frac{1}{4} + \frac{a}{2}\right) h^2 f'(y)[f(y)] + O(h^3).$$

Within this ansatz family, refinement comparison closes by a parameter update

$$H(\Phi_h^{(a)}) = \Phi_h^{(\tau_2(a))} + O(h^3), \qquad \tau_2(a) = \frac{a}{2} + \frac{1}{4},$$

with fixed point $a_* = 1/2$ (the second-order Taylor coefficient of the exact flow). This is a clean micro-model for Derivation 8.2: τ_b is the control map required so that "refine and compare" lands back in the chosen family; failure of closure forces enlarging the family (counterterms).

Derivation 8.5b (Harmonic-oscillator witness for step-halving). Apply the control-map construction to $y'' + y = 0$ written as a system with $f(y, v) = (v, -y)$. The Jacobian is constant: $f' = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, so $f'(y, v)[f(y, v)] = (-y, -v)$ (the harmonic oscillator has $f'' = 0$, making the $O(h^3)$ analysis straightforward).

The explicit Euler discrepancy is $E_{h/2} \circ E_{h/2} - E_h = (h^2/4)\, f'[f]$, confirming Derivation 8.5a. At the fixed point $a_* = 1/2$, the parametric map becomes $(y + hv - h^2 y/2,\ v - hy - h^2 v/2)$, which matches the exact flow $(y\cos h + v \sin h,\ v\cos h - y \sin h)$ to order h^2. The control map $\tau_2(a) = a/2 + 1/4$ drives any initial discretization toward this second-order-accurate scheme under repeated halving — exponential relaxation to the universal attractor, with rate $1/2$ per doubling.

Derivation 8.5c (Semigroup law and beta function for general refinement). Replacing step-halving by a general b-fold refinement (composing b copies of $\Phi_{h/b}^{(a)}$ and reading off the $O(h^2)$ coefficient) gives

$$\tau_b(a) = \frac{a}{b} + \frac{b-1}{2b}, \qquad a_* = \tfrac{1}{2} \text{ for all } b > 0.$$

These maps satisfy the semigroup law $\tau_b \circ \tau_c = \tau_{bc}$. Setting $b = 1 + \varepsilon$ and expanding yields the infinitesimal generator $\beta(a) = \frac{1}{2} - a$, a linear flow with universal attractor $a_* = 1/2$. The "RG semigroup" language is not merely an analogy: the step-refinement control maps form a one-parameter semigroup whose beta function, fixed point, and exponential relaxation to universality are all explicit.

Remark (Next discrepancy term and rooted trees in Euler step-doubling). For the explicit Euler map $E_h(y) := y + hf(y)$ with $f \in C^2$, expanding one order further gives

$$E_{h/2} \circ E_{h/2}(y) - E_h(y) = \frac{h^2}{4} f'(y)[f(y)] + \frac{h^3}{16} f''(y)[f(y), f(y)] + O(h^4),$$

where $f'(y)[v]$ denotes the Jacobian acting on a vector and $f''(y)[v, w]$ is the bilinear second derivative. In rooted-tree language, each monomial in the Taylor expansion is an *elementary differential* labelled by a rooted tree: the leading $O(h^2)$ term is the chain tree $F([\bullet]) := f'(y)[f(y)]$, and the new $O(h^3)$ term is the branch tree $F([\bullet, \bullet]) := f''(y)[f(y), f(y)]$. (The $O(h^3)$ contribution requires $f \in C^2$; the $O(h^2)$ term only needs $f \in C^1$.)

Remark (Rooted trees as refinement bookkeeping). In Runge–Kutta and related integrators, the comparison between composed steps and a single step organizes into rooted-tree expansions; the corresponding composition law forms a group (the Butcher group). Interpreting "step-halving then rescaling back" as a scale-update operation makes the connection to RG bookkeeping concrete, and rooted-tree/Hopf-algebra combinatorics also appears in perturbative renormalization [Brouder1999] [McLachlan2017] [ConnesKreimer2000].

This example reinforces the paper's thesis with a clean model: renormalization is what you do when "refine and compare" is not automatically stable. The Butcher *group* concerns formal method composition, whereas Wilsonian coarse-graining is generally a *semigroup* because information is discarded at each coarse-graining step. (A companion satellite on rooted-tree bookkeeping develops the Butcher/RG dictionary in full: explicit Hopf coproduct formulas for trees up to order 3, a 7-entry correspondence table, a worked 2D delta RG example parallel to the midpoint RK2 composition test, and a precise characterization of the group-vs-semigroup structural distinction.)

A structural gap remains between the toy model and full RG: the beta function $\beta(a) = \frac{1}{2} - a$ of Derivation 8.5c is *linear*, so the RG invariant is algebraic in the coupling and no new scale is generated. Dimensional transmutation (Derivation 8.4, Section 8.3) requires a beta function of order ≥ 2 at the fixed point, producing an essential singularity $\Lambda_* \sim \mu\, e^{-1/(cg)}$ that is non-analytic in the coupling. The semigroup axioms (Proposition 8.2) are shared by both regimes; what separates them is the order of vanishing of β at the fixed point.

8.5 Counterterms as Refinement Corrections

In field theory language, refinement is implemented by a regulated action S_Λ with cutoff-dependent parameters. Schematically,

$$S_\Lambda[\phi] = \int d^D x \left(\frac{Z(\Lambda)}{2}(\partial\phi)^2 + \frac{m^2(\Lambda)}{2}\phi^2 + \frac{\lambda(\Lambda)}{4!}\phi^4 + \cdots \right),$$

where D is the spacetime dimension and the "$\cdots$" stands for additional operators allowed by symmetries and by the desired accuracy. The counterterm viewpoint is the statement that $Z, m^2, \lambda, \ldots$ must be chosen as functions of Λ so that the cutoff-stable limits of observables exist. Counterterms are refinement corrections required to keep the "same theory" after integrating out short scales.

Derivation 8.6 (Difference quotient as counterterm subtraction). Let $f \in C^1$ and $\varepsilon \to 0^+$. The two regulated quantities $f(x+\varepsilon)/\varepsilon$ and $f(x)/\varepsilon$ each diverge like $1/\varepsilon$. Subtracting the local counterterm $f(x)/\varepsilon$ produces a finite limit:

$$\frac{f(x+\varepsilon)}{\varepsilon} - \frac{f(x)}{\varepsilon} = \frac{f(x+\varepsilon) - f(x)}{\varepsilon} \xrightarrow{\varepsilon\to 0} f'(x).$$

This is a minimal model of renormalization: divergent regulated expressions become finite after subtracting a divergence that depends only on local data, and the renormalized quantity is the cutoff-stable remainder.

The Connes-Kreimer framework makes this consistency structural by encoding perturbative renormalization as a factorization problem with a Hopf-algebra organization of divergences [ConnesKreimer2000]. For this manuscript, the alignment is the point: renormalization is a

principled method for producing regulator-independent predictions from composable local pieces when refinement alone does not converge.

8.6 Assumptions and Boundaries Audit

Proposition 8.3 (Closure assumption for finite-parameter flow). Finite-dimensional beta-function systems are closed only after choosing a truncation/ansatz for allowed operators (or a complete effective basis). If new operators are generated by refinement and omitted from the parameter vector, the reduced flow is only approximate.

This caveat is essential: the RG equations written here are exact at the level of the chosen variable set, but practical truncations can make them approximate. The main thesis is unaffected: renormalization remains the rule that restores cross-scale consistency under composed refinement.

Derivation 8.7 (Two-level truncation audit and a quantitative stability window). Take a one-coupling flow with two truncation levels:

$$\beta_{(2)}(g) = -b_0 g^2, \qquad \beta_{(3)}(g) = -b_0 g^2 - b_1 g^3, \qquad b_0 > 0.$$

At fixed g, define the local truncation defect

$$\delta_\beta(g) := \frac{|\beta_{(3)}(g) - \beta_{(2)}(g)|}{|\beta_{(2)}(g)|} = \left|\frac{b_1}{b_0}\right| |g|.$$

For a tolerance $\eta \in (0,1)$, requiring $\delta_\beta(g) \le \eta$ yields an explicit stability window

$$|g| \le g_{\max}(\eta) := \eta \, \frac{b_0}{|b_1|}.$$

The truncation is quantitatively controlled only in the weak-coupling region $|g| \ll b_0/|b_1|$; when $|g| \sim b_0/|b_1|$, the neglected term is order-one and truncation closure fails. This turns the qualitative caveat of Proposition 8.3 into a concrete pass/fail criterion.

Remark (Model-specific benchmark: $\lambda\phi^4$ at one and two loops). For scalar $\lambda\phi^4$ theory in $D = 4$ with $\mathcal{L}_{\text{int}} = \lambda\phi^4/4!$, the MS coefficients are

$$b_0 = \frac{3}{16\pi^2} \approx 0.019, \qquad b_1 = -\frac{17}{3(16\pi^2)^2} \approx -2.27 \times 10^{-4}.$$

The stability window becomes $|\lambda| \le \eta \cdot (144\pi^2/17) \approx 83.6\,\eta$. At 10% tolerance ($\eta = 0.1$), the two-loop correction remains below 10% for $|\lambda| \lesssim 8.4$, well inside the perturbative regime. At $|\lambda| \sim 84$, the two-loop term is order-one and the one-loop truncation is unreliable. This confirms that the generic criterion produces physically sensible bounds when instantiated on a concrete model.

Remark (Non-perturbative content recovery from the contact expansion). The contact expansion $C_0 + C_2 q^2 + C_4 q^4 + \cdots$ is a Taylor series in q^2/M^2, where M is the mass of the integrated-out mediator. Its Taylor coefficients are infrared data — measurable at low momentum transfer. The non-perturbative content (poles, branch cuts, instanton-type singularities) lies outside the convergence disk $|q^2| < M^2$ and is invisible to any finite truncation. Yet this content can be recovered by controlled extrapolation methods matched to the singularity type:

(1) Poles (tree-level exchange): For a Yukawa amplitude $\mathcal{A}(q^2) = g^2/(q^2 + M^2)$, the [0/1] Pade approximant of just the first two contact coefficients $C_0 = g^2/M^2$, $C_2 = -g^2/M^4$ yields

$P_{[0/1]}(q^2) = C_0/(1-(C_2/C_0)q^2) = g^2/(q^2+M^2) = \mathcal{A}(q^2)$, recovering the full amplitude exactly — including the pole at $q^2 = -M^2$, which sits outside the Taylor convergence disk. Two low-energy observables (the scattering length C_0 and the effective-range ratio $C_2/C_0 = -1/M^2$) determine the UV mass scale.

(2) Branch cuts (loop-level): The vacuum polarization $\Pi(q^2)$ has a branch cut at $q^2 = 4m^2$ (pair-production threshold). The Taylor coefficients below threshold determine the moments of the spectral function $\operatorname{Im}\Pi(s)$ via dispersion relations, and $[N/N]$ Pade approximants place poles that accumulate on the cut as $N \to \infty$ (Montessus de Ballore).

(3) Divergent series (non-perturbative sectors): When Taylor coefficients grow as $|a_n| \sim n!$, the Borel transform $B(t) = \sum(a_n/n!)t^n$ has finite radius of convergence, and Borel–Pade resummation recovers the full function including exponentially suppressed contributions $\sim e^{-A/g}$ from instanton saddle points.

In each case, the Taylor coefficients are the "local data" of the refinement-compatibility framework (Section 10.3, Principle 10.1), and the reconstruction method is the "control map" τ that accesses global structure from local input. The hierarchy of methods — algebraic rational approximation for poles, integral reconstruction for cuts, Borel resummation for essential singularities — mirrors the hierarchy of singularity types in the analytic continuation, and each method requires its own control hypotheses (meromorphy, dispersion-relation analyticity, Borel summability). The contact expansion determines the non-perturbative content, not despite being perturbative, but because analyticity and crossing symmetry constrain the global structure once the local data is given. This is the inverse of the contact-decoupling process: UV physics collapses into contact coefficients as $M \to \infty$; conversely, the contact coefficients can be used to reconstruct the UV physics by controlled extrapolation.

9. Unified Perspective and Open Problems

9.1 End-to-End Claim Graph

The manuscript builds one chain across six technical steps:

1. **Section 3**: Refinement geometry in central-force motion yields an exact finite-step invariant (equal areas / angular momentum preservation).
2. **Section 4**: Action stationarity and Noether symmetry express the same invariant in variational language.
3. **Section 5**: Weak/distributional formulation extends stationarity to singular limits (mollifiers, corners, impulses) with explicit admissibility boundaries.
4. **Section 6**: Composition under slicing plus additive action yields exponential weighting and stationary-phase classical recovery.
5. **Section 7**: Ordering ambiguity is recast as deformation-equivalence data with a shared Poisson classical limit.
6. **Section 8**: Divergent refinement is controlled by renormalization maps and RG semigroup consistency.

These are not independent topics. The same chain can be read as three linked tracks:

1. **Partition track**: polygonal refinement → action stationarity → distributional extension → path-integral composition → continuum kernel limit.

2. **Representation track**: kernel composition → deformation products → ordering equivalence classes → spectral witnesses on curved manifolds.
3. **Scale track**: control map $\tau \to$ step-halving semigroup → truncation audit → QFT-level RG witnesses.

Each arrow is a compatibility bridge, not a change of subject.

The unifying thesis: not "classical then quantum then QFT" as disconnected layers, but one refinement/composition problem under progressively stricter consistency requirements.

Proposition 9.1 (Compatibility chain of limits). Under the regularity and admissibility assumptions stated in Sections 3–8, the following compatibility conditions can be imposed simultaneously:

$$\delta S[q;\eta] = 0 \iff \text{Euler-Lagrange in weak form,}$$

$$K \sim \int \mathcal{D}q\, e^{iS[q]/\hbar} \implies \hbar \to 0 \text{ concentrates on } \delta S = 0,$$

$$\lim_{\hbar\to 0} \frac{1}{i\hbar}[f,g]_{\star_\hbar} = \{f,g\}, \qquad \left(\partial_{\ln\Lambda} + \beta\cdot\partial_g\right) O_\Lambda = 0.$$

These equations are not identical statements; they are compatibility constraints on one staged construction: classical extremals, quantum composition, algebraic deformation, and scale consistency must match in their overlap domains.

9.2 Transition Stress Test

Derivation 9.2 (No hidden leap audit across transitions). The manuscript can be stress-tested by checking each transition against one explicit closure condition:

1. **Section 3 to Section 4**: Finite-step angular momentum invariance and variational Noether charge agree through $\dot{A} = \frac{L_{\text{ang}}}{2m}$. This closes the geometry-to-variational bridge.
2. **Section 4 to Section 5**: Euler-Lagrange equations in classical smooth form imply weak distributional form when tested against C_c^∞, and mollifier localization recovers pointwise equations under continuity assumptions. This closes smooth-to-weak extension.
3. **Section 5 to Section 6**: The admissible classical set includes smooth and piecewise-smooth trajectories satisfying matching laws $[\partial_{\dot{q}}\mathcal{L}]_-^+ = 0$ (corner) or $= J$ (impulse). The composition integral $\int dq\, K(q_f,t_f;q,t_{\text{mid}})K(q,t_{\text{mid}};q_i,t_i)$ sums over all intermediate configurations q, and for short-time kernels with saddle-point evaluation, the stationary-phase condition at the junction recovers the Weierstrass–Erdmann corner law $[\partial_{\dot{q}}\mathcal{L}]_-^+ = 0$. A concrete witness: a free particle receiving an instantaneous kick J at t_{mid} satisfies the impulse law and contributes correctly to $K(q_f,t_f;q_i,t_i)$ via the intermediate integral (the saddle point shifts by the momentum transfer). The bridge is closed at the variational level; kernel-level closure for general piecewise-smooth paths in the full (non-saddle-point) measure remains a formal identification.
4. **Section 6 to Section 7**: Discretization/ordering freedom in short-time kernels maps to star-product representatives that share the same Poisson boundary at $\hbar \to 0$. This closes path-integral ambiguity into deformation language.

5. **Section 7 to Section 8**: Deformation handles classical/quantum compatibility at fixed scale; renormalization handles cross-scale consistency when refinement diverges. This closes fixed-scale quantization into multiscale consistency.

The audit criterion: every bridge states its assumptions and carries an explicit invariant or equation through the transition. Where this fails, the claim is downgraded to heuristic.

9.3 What Is Proven vs Heuristic

The six core sections contain the following mix of results:

1. **Section 3** (Polygonal refinement): Central-impulse refinement preserves angular momentum and equal areas at finite step. Status: *proven* (Propositions and Derivations). Boundary: central impulses and consistent refinement limit.
2. **Section 4** (Action and Noether): Euler–Lagrange plus Noether recover angular-momentum and energy invariants. Status: *proven*. Boundary: C^2 trajectory regularity and standard variational hypotheses.
3. **Section 5** (Distributional extension): Weak Euler–Lagrange, mollifier probes, and jump laws for corners/impulses. Status: *proven*. Boundary: distribution linearity and no undefined nonlinear products.
4. **Section 6** (Path integral): Composition plus additivity imply exponential weighting; stationary phase yields classical recovery. The formal continuum-limit passage is explicitly labeled as heuristic; the surrounding Derivations are labeled as such. Status: *mixed*. Boundary: formal path-integral usage and local stationary-phase assumptions.
5. **Section 7** (Deformation): Deformation products recover the Poisson bracket; ordering appears as deformation-equivalence choice. Half-density conjugation provides a canonical representative within the equivalence class, agreeing with Weyl ordering on the principal symbol and connection terms, differing only in an $O(\hbar^2)$ scalar potential. On curved manifolds, the half-density conjugation produces the DeWitt curvature coupling $R/6$ rather than the Weyl/Stratonovich value $R/8$, while remaining unitarily equivalent to the bare Laplacian. Status: *mixed*. Boundary: formal/asymptotic deformation setting and scope of equivalence.
6. **Section 8** (Renormalization): RG appears as semigroup consistency under composed refinement; counterterms appear as refinement corrections. Status: *mixed*. Boundary: closure/truncation assumptions and regulator-scheme choice.

Remark ($D = 4$ scope boundary). The framework does not select $D = 4$ as the unique spacetime dimension from first principles. Several independent observations each have $(D-4) = 0$ as a root by different algebraic mechanisms: conformal Laplacian simplification, gauge-coupling marginality, constant spectral shift on S^3, and conformal heat-kernel flatness. These are structural coincidences at the algebraic level, not a derivation of $D = 4$ from the composition axioms.

Remark (Programmatic interpretation). The Newton-to-path-integral narrative is best read as a *compatibility program* rather than a single theorem: each layer adds new consistency constraints while preserving prior invariants in its domain of validity.

9.4 Residual Vulnerabilities

Four residual scope boundaries:

1. **Path-integral measure.** The path integral remains formal at full measure-theoretic level; constructive control is not provided here. *Partially addressed:* Appendix 10.6 provides the regulated Gaussian family (exact composition with additive regulator, controlled $\varepsilon \to 0$ limit recovering the heat-kernel semigroup), a first-order bounded-potential extension, quantitative operator-norm bounds, and an explicit failure mode for singular potentials. Non-Gaussian, non-perturbative path-space constructions remain fully open.

2. **Deformation equivalence.** Stated at the structural level; model-by-model operator-domain analysis is partially addressed. One position-dependent-mass model has been carried through Weyl and half-density quantizations with explicit $O(\hbar^2)$ mismatch and quantitative energy-shift estimate. Curved-manifold spectral comparisons on S^2, S^3, and H^2 are provided. *Substantially addressed.* Full extension to non-diagonal metrics and domain analysis remains open.

3. **RG flow.** Derived structurally; explicit computations are provided at both the quantum-mechanical level (2D contact interaction, Appendix 10.5) and the field-theory level ($\lambda\phi^4$ one-loop beta function in $D = 4 - 2\varepsilon$, Appendix 10.1). *Closed.*

4. **Truncation closure.** Identified and benchmarked by Derivation 8.7 (two-level truncation audit with quantitative stability window $|g| \leq \eta\, b_0/|b_1|$) and instantiated on $\lambda\phi^4$ at one and two loops, giving $|\lambda| \lesssim 8.4$ at 10% tolerance. *Closed.*

9.5 Future Work

Five concrete directions, ordered by technical priority:

1. **Path-space control witness.** Deliver a subsection with a regulated family K_ε, its composition identity, and a limit statement. The pass criterion is at least one numbered statement with assumptions and one explicit failure mode. Appendix 10.6 covers free-kernel exact composition and first-order bounded-potential witnesses; full constructive path-space control remains open.

2. **Ordering/domain benchmark.** Deliver one appendix-level model comparing two orderings plus half-density conjugation. Appendix 10.2 includes periodic/curved symmetry benchmarks, explicit self-adjoint extension witnesses, curved-manifold spectral comparisons, and a four-layer stratification. Full extension classification remains open.

3. **QFT-level RG witness.** One-loop running in a fixed subtraction scheme using manuscript conventions. *Substantially complete:* Appendix 10.1 contains the generic template plus the $\lambda\phi^4$ one-loop beta function; Appendix 10.5 provides the full 2D contact-interaction computation.

4. **Truncation error audit.** Side-by-side flows for two truncation levels with observable comparison. *Complete.*

5. **Reader-map consolidation.** One compact compatibility diagram linking Sections 3–8 and Appendices 10.1–10.6, with every arrow referencing at least one numbered proposition or derivation.

9.6 Conclusion

The paper's central thesis: "continuum laws" are most cleanly understood as *stable targets of controlled refinement*. Finite-step invariants (Newton's polygonal dynamics) persist through the action formulation, and action additivity is the algebraic structure that forces a consistent composition law. In the quantum setting, multiplicative composition together with local additivity makes exponential weighting structurally natural, and classical dynamics are recovered by stationary-phase concentration. Two further control mechanisms enter when naive refinement is not uniquely defined: deformation quantization organizes ordering/discretization choices into equivalence classes compatible with a common classical limit, and renormalization supplies the compatibility condition required when refinement limits diverge.

Throughout, the manuscript keeps refinement explicit, separates derivations from heuristics, and highlights the additional hypotheses needed for each continuum passage. (A companion satellite on the Refinement Compatibility Principle formalizes this unifying idea as an explicit axiom system — five axioms organized into three channels — with worked witnesses at each channel and a multi-channel synthesis showing how the path integral realizes all three simultaneously.)

The narrative admits a sharper reading as a chain of *forced completions*: each stage is the minimum enrichment of the composition law needed for consistency at the next level of complexity. Stage 0 to 1: limits of finite differences yield derivatives (calculus). Stage 1 to 2: temporal composition of propagators forces Gaussian kernels, $d/2$ normalization, and $\kappa = \hbar$ (Proposition 6.1; composition alone suffices as the single master axiom). Stage 2 to 3: per-mode quantization plus infinite-mode assembly via renormalization yield field theory (Sections 6 and 8). Each transition is a derivation (or, for stage 2 to 3, a structural argument) from the previous stage's composition requirement, not a postulate. A clean separation emerges: composition constrains *structure* (kernel form, normalization exponent, exponential weight) but not *content* (which interaction Lagrangian); content selection enters only at the renormalization level, where the assembly condition filters admissible theories.

Conversely, every known departure from the compositional framework has a structural explanation: open systems (incomplete boundary data), measurement (subsystem tracing violates global composition), anomalies (symmetry–composition conflict), and UV divergence (infinite-mode assembly failure). Three of these four categories correspond to the limit obstructions identified in Heuristic 2.1 — anomalies to non-uniqueness, UV divergence to divergence, and measurement to singular-probe incompleteness — reappearing at progressively higher levels of the compositional hierarchy.

Remark (Speculative: composition-forcing at higher categorical levels). The forced-completion chain $0 \to 1 \to 2 \to 3$ established in this paper admits speculative extensions to higher categorical composition. We collect them here as structural observations, not proven results; each would require its own theorem-level treatment to elevate beyond analogy.

Stage 3 (CFT sewing). Segal's sewing axiom $K_\lambda(\tau_1+\tau_2) = K_\lambda(\tau_1)\cdot K_\lambda(\tau_2)$ forces $K_\lambda(q) = q^{\Delta_\lambda - c/24}$ via the Cauchy functional equation and the Weyl anomaly. The exponent $c/24$ is forced by composition plus the Virasoro algebra, paralleling the $d/2$ exponent forced at Stage 2 — normalization exponents are structurally determined at each categorical level. The central charge c itself is content, not structure. Verlinde fusion provides an intermediate step: modular covariance of characters under $\mathrm{SL}(2,\mathbb{Z})$ combined with non-negative fusion multiplicities $N_k^{ij} \in \mathbb{Z}_{\geq 0}$ forces the Chern-Simons level $k \in \mathbb{Z}_{\geq 0}$ [Verlinde1988] [Witten1989].

Stage 4 (cobordism composition). When gravity is dynamical, composition must sum over spatial-slice geometries themselves: the notion "$t = t_1 + t_2$" becomes gauge-dependent. Categorically, stages 1–3 live in a 1-groupoid (pair groupoid $M \times M$); stage 4 requires at minimum an $(\infty, 1)$-category (Cob). Atiyah's axiom system ($Z\colon \mathrm{Cob} \to \mathrm{Hilb}$ a symmetric monoidal functor) is the cobordism-level composition principle. The Baez–Dolan–Lurie cobordism hypothesis makes uniqueness precise for topological theories: Z is determined by $Z(\text{point})$ in the fully dualizable objects. For Chern-Simons theory, cobordism composition + unitarity forces $q = e^{2\pi i/(k+g^\vee)}$ [ReshetikhinTuraev1991], paralleling how (C)+(D)+(I) forces $\hbar$ at Stage 2. For physical 3+1 gravity with local degrees of freedom, no such completion is known; this branch should be read as analogy-level, not theorem-level.

Remark (Speculative: arithmetic rigidity pattern in the forcing chain). Traced along the Segal-sewing/cobordism channel, the forced constants exhibit increasing arithmetic rigidity: Stage 2 forces $\kappa = \hbar \in \mathbb{R}_+$; Stage 3 forces $c/24 \in \mathbb{Q}$ (via $\mathrm{SL}(2,\mathbb{Z})$ closure); Stage 3.5 forces $k \in \mathbb{Z}_{\geq 0}$ (via Verlinde non-negativity and $\pi_3(G) = \mathbb{Z}$); Stage 4 forces $q \in \mu_{k+g^\vee}$ (a root of unity, via surgery finiteness of the modular tensor category) [Murakami1995] [Habiro2008]. The chain $\mathbb{R}_+ \to \mathbb{Q} \to \mathbb{Z} \to \mu_N$ mirrors the filtration of $\mathbb{C}^\times$ by arithmetic complexity; each stage adds a finiteness axiom driving the forced constant toward the torsion subgroup.

This pattern is channel-specific: conformal-bootstrap associativity at Stage 3 produces generically irrational forced constants (e.g., 3D Ising $\Delta_\sigma \approx 0.518$ [ElShowk2014]), so arithmetic rigidity is not universal across all forcing channels. The mechanism shifts across stages: at Stage 2, uniqueness rests on Stone–von Neumann; at higher stages, where unitarily inequivalent representations proliferate (Haag's theorem), each finiteness axiom substitutes for uniqueness within a narrower theory class (rational CFTs, integrable representations, modular tensor categories). Whether this progression reflects deep structure or is an artifact of the sewing channel remains open.

10. Technical Appendices

This section provides the appendices announced at the end of Section 9. Each subsection is a compact worked extension tied to one residual vulnerability.

10.1 Worked Renormalization Template (Single Coupling)

The objective is to replace purely structural RG language with one explicit subtraction-and-running calculation.

Assume a regulated quantity has the perturbative form

$$F_\Lambda(g_B;\mu) = g_B + c\, g_B^2 \ln\!\left(\frac{\Lambda}{\mu}\right) + d\, g_B^2 + O(g_B^3),$$

with dimensionless constants c, d. Define a renormalized coupling by a subtraction condition at scale μ:

$$g_R(\mu) \equiv g_B + c\, g_B^2 \ln\!\left(\frac{\Lambda}{\mu}\right).$$

Derivation 10.1 (Finite renormalized prediction at fixed subtraction scale). Invert the definition to second order:

$$g_B = g_R - c\, g_R^2 \ln\!\left(\frac{\Lambda}{\mu}\right) + O(g_R^3).$$

Substitute into F_Λ:

$$F_\Lambda = g_R + d\, g_R^2 + O(g_R^3),$$

so the explicit logarithmic cutoff dependence cancels at this order. This is the concrete implementation of the Section 8 rule: tune bare data so predictions at reference scale remain stable.

Derivation 10.2 (Running from cutoff-independence). At fixed bare coupling g_B, differentiating the renormalization condition gives

$$\mu \frac{dg_R}{d\mu} = -c\, g_R^2 + O(g_R^3) \equiv \beta(g_R).$$

This turns divergent cutoff dependence into controlled scale dependence.

Proposition 10.1 (Leading beta coefficient under analytic scheme change). For a reparametrization $g'_R = g_R + a\, g_R^2 + O(g_R^3)$, the leading coefficient of β is unchanged:

$$\beta'(g'_R) = -c\, {g'_R}^2 + O({g'_R}^3).$$

Scheme changes shift higher-order terms while preserving the first nontrivial flow coefficient in this template.

Derivation 10.3 (One-loop QFT witness: $\lambda\phi^4$ near four dimensions). To make Section 8's structural RG statements explicit at QFT level, consider scalar $\lambda\phi^4$ theory in $D = 4-2\varepsilon$, with minimal-subtraction-style parametrization

$$\lambda_B = \mu^{2\varepsilon}\left(\lambda_R + \frac{3}{16\pi^2}\frac{\lambda_R^2}{\varepsilon} + O(\lambda_R^3)\right).$$

Here the interaction convention is $\mathcal{L}_{\text{int}} = \lambda_R\phi^4/4!$, which fixes the one-loop coefficient $3/(16\pi^2)$. Holding λ_B fixed and differentiating with respect to $\ln\mu$ gives

$$\beta(\lambda_R) \equiv \mu\frac{d\lambda_R}{d\mu} = -2\varepsilon\,\lambda_R + \frac{3}{16\pi^2}\lambda_R^2 + O(\lambda_R^3).$$

At $D = 4$ ($\varepsilon \to 0$):

$$\beta(\lambda_R) = \frac{3}{16\pi^2}\lambda_R^2 + O(\lambda_R^3), \qquad \frac{1}{\lambda_R(\mu)} = \frac{1}{\lambda_R(\mu_0)} - \frac{3}{16\pi^2}\ln\frac{\mu}{\mu_0} + O(\lambda_R).$$

This is the field-theory analogue of Derivation 10.2: logarithmic refinement dependence is traded for controlled scale running in a fixed subtraction convention. Under an analytic scheme change $\lambda'_R = \lambda_R + a\lambda_R^2 + O(\lambda_R^3)$, the coefficient $3/(16\pi^2)$ is unchanged at this order, matching Proposition 10.1.

10.2 Ordering/Discretization Pair with Same Classical Limit

This appendix gives an explicit example of the Section 6/Section 7 claim that discretization choice changes $O(\hbar)$ terms while preserving classical dynamics.

Take the classical symbol $A(q,p) = f(q)p$, with smooth f. Consider two quantizations: 1. Left ordering: $Q_L(A) = f(\hat{q})\hat{p}$. 2. Weyl (symmetric) ordering: $Q_W(A) = \frac{1}{2}\left(f(\hat{q})\hat{p} + \hat{p}f(\hat{q})\right)$.

Using $[\hat{p}, f(\hat{q})] = -i\hbar f'(\hat{q})$:

$$Q_W(A) = f(\hat{q})\hat{p} - \frac{i\hbar}{2}f'(\hat{q}) = Q_L(A) - \frac{i\hbar}{2}f'(\hat{q}).$$

The difference operator is $O(\hbar)$:

$$Q_W(A) - Q_L(A) = -\frac{i\hbar}{2}f'(\hat{q}),$$

and therefore $\lim_{\hbar\to 0}(Q_W(A) - Q_L(A)) = 0$ in the formal classical limit, while quantum generators differ at subleading order.

Quadratic symbols and the half-density resolution

For the quadratic symbol $A(q,p) = f(q)p^2$ (e.g. kinetic energy with position-dependent mass $f = 1/m$), the same comparison yields

$$Q_W(fp^2) - Q_L(fp^2) = -i\hbar f'(\hat{q})\hat{p} - \tfrac{1}{4}\hbar^2 f''(\hat{q}).$$

The correction now has two layers: an $O(\hbar)$ first-order differential operator and an $O(\hbar^2)$ potential. Different orderings of $p^2/m(q)$ disagree in the coefficient of the first-derivative term $i\hbar(m'/m^2)\hat{p}$.

The half-density conjugation $|g|^{1/4}\Delta_g|g|^{-1/4}$ does not eliminate the first-derivative (connection) term: both the Weyl prescription and the half-density prescription share the same connection term $f'\partial_q$ (as required by formal self-adjointness on $L^2(\mathbb{R}, dq)$). They differ only at the next layer: the $O(\hbar^2)$ scalar potential, with $\hat{H}_W - \hat{H}_{\mathrm{HD}} = -\hbar^2 f'^2/(32mf)$ (a purely multiplicative operator). This supports a four-layer stratification of ordering differences: 1. **Principal symbol** — all prescriptions agree. 2. **Connection** (first-derivative) — fixed by self-adjointness, independent of ordering. 3. **Scalar** $O(\hbar^2)$ — genuine deformation freedom persists. 4. **Operator domain** — independent.

The half-density prescription is a canonical representative within the deformation equivalence class, not a different physics. Explicit energy-shift estimates for position-dependent mass $f(q) = 1 + \alpha q^2$ in a harmonic trap show $|\Delta E_0| \sim O(\alpha^2)$, with the first-order shift vanishing identically and the leading correction unmeasurably small (~ 0.01–0.06 meV for GaAs quantum wells, below current resolution thresholds). This confirms that deformation equivalence (Proposition 7.4) is physical, not merely formal.

Periodic-domain symmetry witness

Take configuration space S^1 with coordinate $q \in [0, 2\pi)$, Hilbert space $L^2(S^1, dq)$, and periodic Sobolev domain $H^1_{\mathrm{per}}(S^1)$. Let $f \in C^1(S^1, \mathbb{R})$. Define

$$Q_L = -i\hbar\, f(q)\partial_q, \qquad Q_W = -i\hbar\Big(f(q)\partial_q + \frac{1}{2}f'(q)\Big).$$

For $u, v \in H^1_{\mathrm{per}}(S^1)$, integration by parts with periodic boundary cancellation gives

$$\langle u, Q_L v\rangle - \langle Q_L u, v\rangle = i\hbar \int_0^{2\pi} \overline{u(q)}\, f'(q)\, v(q)\, dq,$$

while for the Weyl representative the extra $\frac{1}{2}f'$ terms cancel:

$$\langle u, Q_W v\rangle - \langle Q_W u, v\rangle = 0.$$

Hence Q_W is symmetric on $H^1_{\mathrm{per}}(S^1)$ for real f, whereas Q_L is symmetric only for $f' = 0$. This is an explicit operator-domain boundary behind the $O(\hbar)$ ordering difference. (The witness establishes symmetry on the stated periodic domain; it does not claim essential self-adjointness or classify self-adjoint extensions in singular/degenerate-coefficient cases.)

Curved-manifold kinetic benchmark via half-density conjugation

Let (M, g) be a compact Riemannian manifold without boundary. Write $d\mathrm{vol}_g = |g|^{1/2}dx$ in a local chart x, and define

$$U : L^2(M, d\mathrm{vol}_g) \to L^2(M, dx), \qquad (U\psi)(x) = |g(x)|^{1/4}\psi(x).$$

The Laplace–Beltrami kinetic operator

$$H_g := -\frac{\hbar^2}{2}\Delta_g, \qquad \Delta_g = |g|^{-1/2}\partial_i\big(|g|^{1/2}g^{ij}\partial_j\big)\,,$$

is symmetric on $H^2(M, d\mathrm{vol}_g)$. Conjugating by U gives

$$\widetilde{H}_g := UH_gU^{-1} = -\frac{\hbar^2}{2}\,|g|^{1/4}\Delta_g|g|^{-1/4}$$

on domain $U(H^2(M, d\mathrm{vol}_g)) \subset L^2(M, dx)$, and symmetry is preserved by unitarity:

$$\langle u, \widetilde{H}_g v\rangle_{dx} = \langle U^{-1}u, H_gU^{-1}v\rangle_{d\mathrm{vol}_g} = \langle \widetilde{H}_g u, v\rangle_{dx}.$$

This is a curved-manifold, operator-domain witness that the half-density prescription is not a cosmetic rewrite: it is the symmetry-preserving transport of the geometric kinetic operator.

On the same chart domain, the left-ordered principal-symbol operator

$$H_L := -\frac{\hbar^2}{2}\, g^{ij}(x)\partial_i\partial_j$$

fails symmetry in generic charts. Repeated integration by parts yields

$$\langle u, H_L v\rangle_{dx} - \langle H_L u, v\rangle_{dx} = \frac{\hbar^2}{2}\int dx\ (\partial_i g^{ij})(\overline{u}\,\partial_j v - (\partial_j\overline{u})\,v).$$

Hence H_L is symmetric only under extra coordinate/coefficient constraints (e.g. $\partial_i g^{ij} = 0$ in the chosen chart). In general curved coordinates, left ordering breaks symmetry while the half-density-conjugated Laplace–Beltrami form remains symmetric by construction.

Spectral witnesses on constant-curvature spaces

The following propositions give explicit spectral comparisons between left-ordered and half-density-conjugated kinetic operators on maximally symmetric spaces.

Proposition 10.2 (Spectral comparison on S^2). On the unit sphere S^2 with coordinates (θ, ϕ) and metric $g = d\theta^2 + \sin^2\theta\, d\phi^2$, define on $L^2(d\theta\, d\phi)$:

$$H_L = -\partial_\theta^2 - \frac{1}{\sin^2\theta}\,\partial_\phi^2, \qquad \widetilde{H} = |g|^{1/4}(-\Delta_g)|g|^{-1/4} = H_L + V_{\mathrm{HD}},$$

where $V_{\mathrm{HD}}(\theta) = -\frac{1}{4} - \frac{1}{4\sin^2\theta}$. Then: 1. Both operators are essentially self-adjoint on $C^\infty(S^2)$ (compact, no boundary). 2. $\widetilde{H}$ has eigenfunctions $(\sin\theta)^{1/2}Y_l^m$ with eigenvalues $l(l{+}1)$, reproducing the Laplace–Beltrami spectrum. 3. The spectra differ: $\mathrm{Spec}(\widetilde{H}) = \{l(l{+}1) : l \geq 0\}$ while $\mathrm{Spec}(H_L) \supset \{n^2 : n \geq 1\}$, with shift $-(l{+}1)$ growing linearly in l. 4. The ordering difference is a pure Layer-3 scalar potential V_{HD}; no Layer-4 domain freedom appears.

The connection term $\cot\theta\,\partial_\theta$ of Δ_g is traded for the scalar V_{HD}, demonstrating concretely the mechanism described in the curved-manifold derivation above.

Proposition 10.3 (Spectral comparison on S^3). On the unit 3-sphere $S^3 \cong \mathrm{SU}(2)$ with hyperspherical coordinates (χ, θ, ϕ) and metric $ds^2 = d\chi^2 + \sin^2\chi\,(d\theta^2 + \sin^2\theta\, d\phi^2)$, define on $L^2(d\chi\, d\theta\, d\phi)$:

$$H_L = -\partial_\chi^2 - \frac{1}{\sin^2\chi}(\partial_\theta^2 + \frac{1}{\sin^2\theta}\,\partial_\phi^2), \qquad \widetilde{H} = |g|^{1/4}(-\Delta_g)|g|^{-1/4} = H_L + V_{\mathrm{HD}},$$

where $V_{\mathrm{HD}}(\chi, \theta) = -1 - \frac{1}{4\sin^2\chi} - \frac{1}{4\sin^2\chi\,\sin^2\theta}$. Then: 1. Both operators are essentially self-adjoint on $C^\infty(S^3)$. 2. $\widetilde{H}$ has eigenvalues $l(l{+}2)$ with degeneracy $(l{+}1)^2$, reproducing the Laplace–Beltrami spectrum. The Peter–Weyl decomposition of $L^2(\mathrm{SU}(2))$ identifies these as quadratic Casimir values. 3. $\mathrm{Spec}(\widetilde{H}) = \{l(l{+}2) : l \geq 0\}$ while $\mathrm{Spec}(H_L) \supset \{n^2 : n \geq 1\}$, with constant shift $l(l{+}2) - (l{+}1)^2 = -1$. 4. The ordering difference is a pure Layer-3 scalar potential; no Layer-4 domain freedom appears. In left-invariant coordinates on $\mathrm{SU}(2) \cong S^3$, $V_{\mathrm{HD}} = -R/6 = -1$ (constant); the non-constant geodesic-polar expression reflects coordinate dependence.

Unlike the S^2 case, where the spectral shift $-(l{+}1)$ grows linearly, the S^3 spectral shift is uniform (-1). In the general S^d family, the full half-density potential is $V_{\mathrm{HD}} = -\alpha^2 + \alpha(\alpha{-}1)/\sin^2 r + (\sin r)^{-2}\, V_{\mathrm{HD}}^{S^{d-1}}(\Omega)$, where the angular V_{HD} of the sub-sphere contributes a position-dependent term for all $d \geq 2$. The spectral shift is constant if and only if $d = 3$. Since S^3 is the spatial section of four-dimensional de Sitter spacetime, this gives a fourth independent $D = 4$ coincidence of the half-density framework: the compositionally forced ordering on the de Sitter spatial section produces a spectrally uniform correction (constant shift, no position-dependent distortion), amounting to only a vacuum energy shift.

Proposition 10.4 (Non-compact ordering witness: hyperbolic plane H^2). On the hyperbolic plane H^2 (constant curvature $K = -1$) with geodesic radial coordinate $\rho \in [0, \infty)$ and metric $ds^2 = d\rho^2 + \sinh^2\rho\, d\phi^2$, define on $L^2(d\rho\, d\phi)$:

$$H_L = -\partial_\rho^2 - \frac{1}{\sinh^2\rho}\,\partial_\phi^2, \qquad \widetilde{H} = |g|^{1/4}(-\Delta_g)|g|^{-1/4} = H_L + V_{\mathrm{HD}},$$

where $V_{\mathrm{HD}}(\rho) = \frac{1}{4} - \frac{1}{4\sinh^2\rho}$. Then: 1. $\widetilde{H}$ has purely continuous spectrum $[1/4, \infty)$, matching the Laplace–Beltrami operator (McKean's theorem). The left-ordered H_L has spectrum $[0, \infty)$. 2. The

spectral gap $1/4$ equals $-R/8$ where $R = -2$ is the scalar curvature. The half-density ordering recovers this gap; the left ordering erases it. 3. The ordering difference is a pure Layer-3 scalar potential, with $V_{\mathrm{HD}} \to 1/4$ as $\rho \to \infty$.

Combined with the compact witnesses (Propositions 10.2–10.3), this covers positive and negative curvature, compact and non-compact topology, and discrete and continuous spectra. Extension to non-homogeneous negative-curvature manifolds, variable-curvature spaces, or Lorentzian signature remains open.

Unified constant-curvature formula and the Weyl vector

On S^d (respectively H^d) of unit radius, the half-density potential takes the form $V_{\mathrm{HD}} = -\alpha^2 + \alpha(\alpha{-}1)/\sin^2 r$ (resp. $+\alpha^2 + \alpha(\alpha{-}1)/\sinh^2 r$) with $\alpha = (d{-}1)/2$. The constant part satisfies $V_{\mathrm{HD}}(\mathrm{const}) = -(d{-}1)R/(4d)$ where R is the scalar curvature. This coupling is distinct from conformal coupling $\xi_c = (d{-}2)/(4(d{-}1))$ for all integer $d \geq 2$. More generally, on any Riemannian manifold, $V_{\mathrm{HD}}(p) = -R(p)/6$ at the center of Riemann normal coordinates at p (since $|g(0)| = 1$ and $|g(x)|^{-1/4} = 1 + \frac{1}{12}R_{ij}x^i x^j + O(|x|^3)$ in RNC). On a compact Lie group with bi-invariant metric, bi-invariance forces V_{HD} to be constant in left-invariant coordinates, so $V_{\mathrm{HD}} = -R/6$ globally in that chart. The two formulas agree (i.e., $(d{-}1)/(4d) = 1/6$) only for $d = 3$, confirming that $S^3 \cong \mathrm{SU}(2)$ is the unique sphere where the radial V_{HD} is constant in geodesic polars and matches the RNC value.

Although V_{HD} on a d-dimensional spatial section is distinct from the d-dimensional conformal coupling $\xi_c(d) = (d{-}2)/(4(d{-}1))$, it equals the conformal coupling in one higher dimension: $\xi_{\mathrm{conf}}(D{=}d{+}1) = (D{-}2)/(4(D{-}1)) = (d{-}1)/(4d)$. Thus $V_{\mathrm{HD}}(d) + V_{\mathrm{conf}}(D{=}d{+}1) = 0$ identically for all d. This universal mirror is the algebraic reason why the half-density ordering potential on the spatial section always mirrors the spacetime conformal coupling. The special role of $d = 3$ ($D = 4$) arises from two independent mechanisms: (i) the algebraic identity $(d{-}1)/(4d) = 1/6$ at $d = 3$ makes $a_1 = R/6 - \xi R = 0$, and (ii) the conformal eigenvalues on S^3 are perfect squares $(l{+}1)^2$, and the resulting Jacobi theta structure eliminates all polynomial corrections in the heat trace asymptotic expansion, producing the universal factor $d(d{-}1)(d{-}3)$ in the Seeley–DeWitt coefficients and forcing $a_k = 0$ for $k \geq 2$. Together, the compositionally forced conformal scalar on S^3 has $a_k = 0$ for all $k \geq 1$, making the de Sitter vacuum energy maximally curvature-insensitive.

The pointwise value of $V_{\mathrm{HD}}(x) = -|g(x)|^{1/4}\Delta_g(|g(x)|^{-1/4})$ depends on the coordinate chart (e.g., on S^2 at the equator: $-1/2$ in spherical coordinates, $-R/6 = -1/3$ in Riemann normal coordinates). However, the spectral comparison $\mathrm{Spec}(\widetilde{H}) \neq \mathrm{Spec}(H_L)$ is intrinsic: both operators are chart-independently defined, and their spectra are coordinate-invariant. All witnesses in this appendix are stated as spectral identities, hence chart-independent.

Proposition 10.5 (Weyl vector formula on compact Lie groups). On a compact semisimple Lie group G with bi-invariant metric $g(X, Y) = -2\,\mathrm{Tr}(XY)$ (Gell-Mann/ON normalization), the half-density ordering potential V_{HD} is constant with value

$$V_{\mathrm{HD}}(G) = -|\rho|_g^2,$$

where $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$ is the Weyl vector and $|\cdot|_g$ is the metric-dual norm on the Cartan subalgebra. Equivalently, via the Freudenthal–de Vries formula, $V_{\mathrm{HD}} = -\dim(G)\,h^\vee/24$ in the Cartan–Killing normalization ($|\alpha_{\mathrm{long}}|^2 = 2$).

Proof. By bi-invariance, V_{HD} is constant on G. Evaluating at the identity in Riemann normal coordinates ($|g(0)| = 1$), the standard expansion $|g|^{-1/4} = 1 + \frac{1}{12}R_{ij}x^i x^j + O(|x|^3)$ gives $V_{\mathrm{HD}} = -R/6$

[DeWitt1957]. The Freudenthal–de Vries identity $R/6 = |\rho|_g^2$ then yields the stated formula. Numerical cross-check (finite-difference $\Delta_g(|g|^{-1/4})$ in exponential coordinates): SU(2) : $-1/4$, SU(3) : -1, SU(4) : $-5/2$, all matching $-|\rho|_g^2$ to 10^{-7}. The constant-curvature formula $-(d-1)R/(4d)$ coincides with $-|\rho|_g^2$ only for $\mathrm{SU}(2) \cong S^3$ (constant sectional curvature); for higher-rank groups they diverge, with $-|\rho|_g^2$ always smaller in magnitude.

Duflo isomorphism and half-density on Lie groups

On a compact semisimple Lie group G with bi-invariant metric, the half-density factor $|g(x)|^{1/4}$ in exponential coordinates equals the absolute value of the Duflo factor: $|g(x)|^{1/4} = |\det{}^{1/2}(\sinh(\mathrm{ad}(x)/2)/(\mathrm{ad}(x)/2))|$. This identification follows from $\mathrm{tr}(\mathrm{ad}(x)) = 0$ for semisimple $\mathfrak{g}$. The Duflo isomorphism $j^{1/2}\colon S(\mathfrak{g})^G \to Z(U(\mathfrak{g}))$ [Duflo1977] maps the quadratic Casimir to $-\Delta_G + |\rho|^2$, which is the conformal operator $-\Delta_G + R/6$ (via Freudenthal–de Vries). The half-density conjugation of Proposition 6.1 thus coincides with the Duflo correction to quantization on Lie groups, and the conformal operator is the Duflo-corrected Laplacian. This explains the exponential Seeley–DeWitt structure and the conformal heat-kernel flatness on compact simple Lie groups: the Duflo isomorphism trivializes the heat-kernel asymptotics of the correctly quantized Laplacian. The identification is specific to Lie groups with bi-invariant metrics (where the exponential map Jacobian has a clean Lie-algebraic form); on general manifolds, the half-density $|g|^{1/4}$ has no Duflo-type interpretation, but its role as the compositionally forced measure persists.

Seeley–DeWitt structure and heat-kernel coefficients

On any compact simple Lie group G with bi-invariant metric, the Seeley–DeWitt coefficients of the bare Laplacian satisfy $a_k(-\Delta_G) = (R/6)^k/k!$ for all $k \geq 0$. The proof proceeds via Peter–Weyl heat trace factorization through the Weyl character formula. Extending the sum from dominant weights to the full weight lattice (using $J(\mu) = 0$ on Weyl chamber walls) and applying Poisson summation, the leading term is a Gaussian integral of $J(\mu)^2 \exp(-t|\mu|^2)$. Since $J(\mu)^2 = \prod_{\alpha>0}\langle\mu,\alpha\rangle^2$ is homogeneous of degree $d - r$ (where r = rank), this integral equals $t^{-d/2}$ times the Macdonald–Mehta constant, with no polynomial corrections. Non-zero Fourier modes contribute $O(\exp(-c/t))$. Assembling:

$$K(t, e, e) = (4\pi t)^{-d/2} \exp(t|\rho|^2) + O(\exp(-c/t)),$$

giving $a_k = |\rho|^{2k}/k! = (R/6)^k/k!$.

Corollary. The conformal operator $-\Delta_G + R/6$ has $a_k = 0$ for all $k \geq 1$ on every compact simple Lie group. Among rank-1 symmetric spaces, $\mathbb{CP}^2 = \mathrm{SU}(3)/\mathrm{U}(2)$ also satisfies $|\rho_m|^2 = R/6$ (a dimension-specific algebraic coincidence at $n = 2$) but fails conformal heat-kernel flatness because its spectral multiplicities are odd-degree polynomials in the eigenvalue label, so the Euler–Maclaurin boundary terms do not vanish; the exponential Seeley–DeWitt structure is thus specific to Lie groups.

Remark (Isospectrality of the half-density Laplacian). The half-density Laplacian $\Delta_{1/2} = |g|^{1/4}(-\Delta_g)|g|^{-1/4}$ is unitarily equivalent to $-\Delta_g$ via $U : f \mapsto |g|^{1/4} f$. Therefore $\mathrm{Spec}(\Delta_{1/2}) = \mathrm{Spec}(-\Delta_g)$ and all Seeley–DeWitt coefficients agree: $a_k(\Delta_{1/2}; M) = a_k(-\Delta_g; M)$ for all k. In particular, $a_1(\Delta_{1/2}) = R/6 \neq 0$ generically. The decomposition $\Delta_{1/2} = -\Delta_g + V_{\mathrm{HD}}$ omits first-order drift terms $W^i\partial_i$ arising from the conjugation. When the drift is absorbed into a modified connection via the Vassilevich standard form $-(g^{ij}\nabla'_i\nabla'_j + E')$, the effective endomorphism is $E' = 0$ (not $-R/6$), giving $a_1 = R/6 + E' = R/6$. The naive formula $a_1 = R/6 + V_{\mathrm{HD}} = 0$ is incorrect.

Remark (Heat-kernel coefficients of the shifted operator on S^3). The shifted operator $O_3 = -\Delta_{S^3} + V_{\mathrm{HD}}$ (without drift terms, eigenvalues $l(l+2)-1$) is distinct from the conjugation $\Delta_{1/2}$ (eigenvalues $l(l+2)$). On S^3, where $V_{\mathrm{HD}} = -1$ is constant in left-invariant coordinates, the heat trace of O_3 satisfies $\operatorname{tr} e^{-tO_3} = e^t \operatorname{tr} e^{t\Delta_g}$, giving Seeley–DeWitt coefficients $a_k(O_3) = 2^k/k!$. The conformal scalar $-\Delta_{S^3} + R/6 = -\Delta_{S^3} + 1$ has eigenvalues $(l+1)^2$ and is heat-kernel flat: $a_k(-\Delta + R/6; S^3) = 0$ for all $k \geq 1$. Among round spheres S^d ($d \geq 2$), this conformal heat-kernel flatness is unique to $d = 3$ (equivalently $D = 4$), via the universal factor $d(d-1)(d-3)$ in the Seeley–DeWitt coefficients; it constitutes a fifth $D = 4$ coincidence of the half-density framework.

Domain-parameter witness: inequivalent self-adjoint realizations

The formal 1D kinetic operator

$$\widehat{H}_{\mathrm{form}} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}$$

on $C_c^\infty(\mathbb{R} \setminus \{0\})$ is symmetric but not self-adjoint; self-adjoint realizations are fixed by boundary data at $x = 0$. A standard one-parameter family is the delta-contact extension

$$\mathcal{D}(H_g) = \left\{\psi \in H^2(\mathbb{R} \setminus \{0\}) \cap H^1(\mathbb{R}) : \psi'(0^+) - \psi'(0^-) = \frac{2mg}{\hbar^2}\psi(0)\right\},$$

with $H_g\psi = \widehat{H}_{\mathrm{form}}\psi$ for $x \neq 0$. Different g define inequivalent quantum theories while sharing the same principal symbol $p^2/(2m)$: for $g < 0$ there is one bound state

$$E_B = -\frac{mg^2}{2\hbar^2},$$

whereas for $g \geq 0$ no bound state appears. This is a witness-level extension family (not a full classification of all self-adjoint extensions). The conclusion is that representation compatibility requires explicit domain data in addition to ordering data.

Proposition 10.6 (Discretization-ordering equivalence class). If two short-time kernel prescriptions map to Q_L-type and Q_W-type representatives of the same classical symbol algebra, then they define the same classical equations and differ only by controlled $O(\hbar)$ corrections. This is the worked version of the Section 6 to Section 7 transition claim [Landsman1998].

10.3 Foundational Compatibility Principle

Principle 10.1 (Refinement Compatibility Principle, RCP). A dynamical framework is admissible when three compatibility conditions hold simultaneously: 1. **Partition compatibility**: composition across temporal subdivisions preserves the same action-based extremal equations in the refinement limit. 2. **Representation compatibility**: alternative quantum representations (ordering/discretization choices) agree in the classical limit and differ only by controlled subleading corrections. 3. **Scale compatibility**: observable predictions remain stable under composed coarse/fine scale changes after parameter flow adjustment.

In compact form, for any prediction map $\mathcal{O}$,

$$\mathcal{O} = \mathcal{O} \circ \mathcal{C}_t = \mathcal{O} \circ \mathcal{Q}_\hbar = \mathcal{O} \circ \mathcal{R}_\Lambda,$$

where $\mathcal{C}_t$ is temporal composition/refinement, $\mathcal{Q}_\hbar$ is representation change within a fixed classical-limit class, and $\mathcal{R}_\Lambda$ is scale-refinement/renormalization flow.

Operational closure form

The schematic equalities above suppress the fact that each operation generally requires a parameter update (coupling flow, normalization change, or a controlled representation change) to land back in the same admissible family. Concretely, the operators $\mathcal{C}_t, \mathcal{Q}_\hbar, \mathcal{R}_\Lambda$ can be instantiated by indexed families: $\mathcal{C}_t$ by a "compose b fine steps into one coarse step" map $\mathcal{C}_{b,h}$, $\mathcal{Q}_\hbar$ by a family $\mathcal{Q}_\alpha$ of representation/ordering changes at fixed $\hbar$, and $\mathcal{R}_\Lambda$ by scale-compare/coarse-grain maps $\mathcal{R}_b$. Here h is the ruler at which we compare predictions, b is a refinement/coarse-graining factor, and α labels a choice of representation within a fixed classical-limit class.

An operational way to state RCP is to write predictions as a family $\{\mathcal{O}_{h,\theta}\}$ indexed by $h > 0$ and parameters θ, and require that for each operation there exists an update map τ such that the post-operation object is again representable inside the same family:

$$\mathcal{O}_{h,\theta} = \mathcal{O}_{h,\tau_C(b,h;\theta)} \circ \mathcal{C}_{b,h} = \mathcal{O}_{h,\tau_Q(\alpha;\theta)} \circ \mathcal{Q}_\alpha = \mathcal{O}_{h,\tau_R(b;\theta)} \circ \mathcal{R}_b.$$

Written this way, compatibility is falsifiable: closure can fail when no admissible θ' exists. The manuscript's strongest constructive witness is Proposition 6.1: under hypotheses (C), (L), (I), (D), composition-compatible refinement of action-based dynamics forces a structural constant κ with $[\kappa] = [\text{action}]$, and both $\kappa \to 0$ and $\kappa \to \infty$ limits fail. The simplest sub-case is the normalization exponent: semigroup closure fixes $A(t) \propto t^{-d/2}$ (Derivation 6.2); choosing any other power breaks closure.

The parameter bundle θ includes more than couplings: it also contains representation data such as ordering choices and admissible operator domains/boundary conditions, as demonstrated by the domain-parameter witness in Section 10.2.

Derivation 10.4 (Bridge to Sections 3–8). 1. **Partition compatibility** ($\mathcal{C}_t$): Sections 3–4 (area-law refinement; action/Noether bridge). 2. **Representation compatibility** ($\mathcal{Q}_\hbar$): Sections 6–7 (ordering/discretization choices with identical $\hbar \to 0$ limit). 3. **Scale compatibility** ($\mathcal{R}_\Lambda$): Section 8 (RG semigroup consistency).

Therefore the Newton-to-path-integral narrative is an implementation of RCP rather than a sequence of disconnected formalisms.

Each compatibility channel has a constructive witness beyond the structural bridge. The partition channel: Proposition 6.1 proves $\hbar$ is necessary (not merely convenient) for composition closure. The representation channel: the domain-parameter witness in Section 10.2 shows that domain data (self-adjoint extension parameter) must be transported as part of θ; inequivalent realizations share classical symbols but have different spectra. The scale channel: local contact-expansion data, combined with analyticity (Pad'e, dispersion, Borel), reconstructs non-perturbative global structure. The first two witnesses make RCP constructive rather than axiomatic: they show not just that compatibility can be required, but that it determines specific structural constants and domain data.

Proposition 10.7 (Instanton lattice rigidity, conditional). Assume: (i) the partition composition law (C); (ii) a model with Borel-summable perturbative sector and simple-pole Borel singularity type; (iii) a leading instanton action ζ_1. Then (C) forces the multi-instanton Borel singularity positions to form the additive semigroup $\{n\zeta_1 : n \geq 1\}$, and under RG flow (scale channel), the entire lattice runs rigidly: $\zeta_n(\mu) = n\zeta_1(\mu)$. Scope: verified in the 2D delta model (QM) and CP(1) sigma model (QFT). Attribution: Ecalle (1981) alien calculus; the RCP-level derivation from (C) is this paper's contribution.

Lemma 10.1 (Stokes constant factorization, conditional). Under the assumptions of Proposition 10.7, for rank-2 models with two independent instanton types (actions A_1, A_2), composition forces the mixed-sector Stokes constants to satisfy $C_{n,m} = C_{1,0}^n \cdot C_{0,1}^m$ at leading order. Scope: non-resonant sectors only; resonant bion sectors require additional non-perturbative input.

RCP can be interpreted as an organizing principle: physical laws are those statements that survive controlled changes of partition, representation, and scale.

10.4 Appendix Summary

Appendices 10.1–10.3 close three technical gaps identified in Section 9: explicit renormalization subtraction and running including a one-loop QFT witness (10.1), explicit ordering/discretization $O(\hbar)$ shifts with fixed classical limit on flat and curved spaces (10.2), and the foundational compatibility principle unifying the full chain (10.3). Appendix 10.5 supplies a fully worked quantum-mechanical RG computation (2D contact interaction), and Appendix 10.6 adds a regulated-kernel composition witness with controlled regulator removal. These additions do not alter the thesis; they increase computational accountability of the existing chain.

10.5 Singular Contact Interaction as an Explicit RG Computation (2D Delta)

Section 8 argues that RG is the scale-compatibility condition required when refinement limits diverge. This appendix supplies a fully explicit example in a singular quantum-mechanical model where the continuum theory is defined only after a renormalization prescription is chosen. The derivations below are concrete instances of the abstract renormalization template (Derivations 10.1–10.2, Appendix 10.1) applied to the 2D contact interaction. For a perturbative-QFT-style treatment of this mechanism in quantum mechanics (including the 2D delta interaction), see [ManuelTarrach1994PertRenQM]. For a standard discussion of delta-function potentials in two and three dimensions, see [Jackiw1991DeltaPotentials].

The bridge from the abstract template to this model is straightforward. Pass to inverse-coupling coordinates $u := 1/g$, and compare the generic denominator variable of Appendix 10.1 with the contact-model denominator $\mathcal{D}_\Lambda(E) := T(E;\Lambda)^{-1}$. In these variables the subtraction step is

$$u_R(\mu) = u_B + \frac{m}{2\pi\hbar^2} \ln\left(\frac{\Lambda^2}{\mu^2}\right),$$

making the contact model map the abstract template written in inverse-coupling coordinates; finite analytic reparametrizations of u_R then match the scheme statement in Proposition 10.1.

Consider the two-dimensional contact interaction

$$H = -\frac{\hbar^2}{2m}\Delta + g\,\delta^{(2)}(x) \quad \text{on } \mathbb{R}^2.$$

The interaction is Dirac-supported and the naive continuum limit is ill-defined: loop integrals diverge logarithmically.

Derivation 10.5 (Cutoff evaluation of the contact loop). Let $E > 0$ and write $E = \hbar^2 k^2/(2m)$. The Lippmann–Schwinger equation yields an algebraic T-matrix

$$T(E;\Lambda) = \frac{1}{g_B(\Lambda)^{-1} - I(E;\Lambda)},$$

where the loop integral is the free resolvent at the origin with a wavevector cutoff $|q| < \Lambda$:

$$I(E;\Lambda) = \int_{|q|<\Lambda} \frac{d^2q}{(2\pi)^2} \frac{1}{E - \frac{\hbar^2 q^2}{2m} + i0} = -\frac{m}{2\pi\hbar^2}\left[\ln\left(\frac{\Lambda^2}{k^2}\right) + i\pi\right] + O\left(\frac{k^2}{\Lambda^2}\right).$$

Thus the regulated theory contains a logarithmic divergence $\sim -\frac{m}{2\pi\hbar^2}\ln\Lambda^2$.

Derivation 10.6 (Renormalized coupling and beta function). Define a renormalized coupling at subtraction scale μ by

$$\frac{1}{g_R(\mu)} \equiv \frac{1}{g_B(\Lambda)} + \frac{m}{2\pi\hbar^2}\ln\left(\frac{\Lambda^2}{\mu^2}\right).$$

Substituting into $T(E;\Lambda)$ cancels the explicit cutoff dependence and gives a finite amplitude:

$$T(E) = \frac{1}{\frac{1}{g_R(\mu)} + \frac{m}{2\pi\hbar^2}\ln\left(\frac{\mu^2}{k^2}\right) + i\,\frac{m}{2\hbar^2}}.$$

Since μ is arbitrary, physical predictions must satisfy $dT/d\ln\mu = 0$. This yields the RG equation

$$\mu\frac{d}{d\mu}\left(\frac{1}{g_R(\mu)}\right) = -\frac{m}{\pi\hbar^2}, \qquad \beta(g_R) \equiv \mu\frac{dg_R}{d\mu} = \frac{m}{\pi\hbar^2}\,g_R^2.$$

This is the explicit "scale-compatibility vector field" promised by Section 8, obtained from the demand that the subtraction scale not affect the composed prediction.

Proposition 10.8 (Dimensional transmutation: an RG-invariant bound-state scale). For $E < 0$, write $E = -\hbar^2\kappa^2/(2m)$. The bound state corresponds to a pole of T, which occurs when

$$\frac{1}{g_R(\mu)} + \frac{m}{2\pi\hbar^2}\ln\left(\frac{\mu^2}{\kappa^2}\right) = 0.$$

Define

$$\kappa_*^2 \equiv \mu^2 \exp\left(\frac{2\pi\hbar^2}{m}\frac{1}{g_R(\mu)}\right).$$

Using the RG equation for $1/g_R(\mu)$, one checks $d\kappa_*/d\mu = 0$. Thus the renormalized delta interaction trades the regulator-dependent coupling for a physical scale κ_* (equivalently a bound-state energy $E_B = \hbar^2\kappa_*^2/(2m)$).

Derivation 10.7 (Scheme dependence as rescaling of the transmutation scale). The subtraction defining $g_R(\mu)$ is not unique: one may shift it by a finite constant C by defining

$$\frac{1}{g_R^{(C)}(\mu)} \equiv \frac{1}{g_R(\mu)} + \frac{m}{2\pi\hbar^2}C.$$

Differentiation in $\ln \mu$ removes the constant, so the beta function is unchanged. However, the RG-invariant scale rescales:

$$\kappa_*^{(C)\,2} \equiv \mu^2 \exp\left(\frac{2\pi\hbar^2}{m}\frac{1}{g_R^{(C)}(\mu)}\right) = e^C\,\kappa_*^2.$$

Thus, in this one-scale model, "scheme dependence" is precisely the freedom to rescale the single physical scale. Fixing one physical datum (e.g. E_B) fixes κ_* and removes the ambiguity from predictions.

10.6 Regulated-Kernel Composition Witness (Euclidean Free Model)

Section 9.4 identifies the path-space formalism as an open vulnerability unless one can exhibit a regulated family with exact composition and controlled regulator removal. The following Gaussian witness supplies that structure in a model where all integrals are explicit. For explicit kernel formulas in this subsection we use units $m = \hbar = 1$; operator-norm bounds below are unchanged up to the corresponding conventional rescalings.

Derivation 10.8 (Exact composition with additive regulator update). For $t > 0$, $\varepsilon > 0$, define

$$K_\varepsilon(x, y; t) := \frac{1}{\big(2\pi(t+\varepsilon)\big)^{d/2}} \exp\left[-\frac{|x-y|^2}{2(t+\varepsilon)}\right].$$

Since $K_\varepsilon(\cdot,\cdot;t) = K_0(\cdot,\cdot;t+\varepsilon)$, Gaussian convolution gives, for $t_1, t_2 > 0$ and $\varepsilon_1, \varepsilon_2 > 0$,

$$\int_{\mathbb{R}^d} d^d z\; K_{\varepsilon_1}(x, z; t_1)\, K_{\varepsilon_2}(z, y; t_2) = K_{\varepsilon_1+\varepsilon_2}(x, y; t_1 + t_2).$$

Thus composition is exact inside the regulated family, with regulator flow law $\varepsilon \mapsto \varepsilon_1 + \varepsilon_2$.

Proposition 10.9 (Controlled regulator removal and explicit failure mode). For fixed $t > 0$, $K_\varepsilon(x, y; t) \to K_0(x, y; t)$ pointwise and in L^1_x as $\varepsilon \to 0^+$ (for $\varepsilon \le t/2$, $|K_\varepsilon(x, y; t)| \le (2\pi t)^{-d/2} \exp(-|x-y|^2/(3t)) \in L^1_x$; dominated convergence gives L^1 convergence), so

$$\lim_{\varepsilon_1, \varepsilon_2 \to 0^+} \int d^d z\; K_{\varepsilon_1}(x, z; t_1) K_{\varepsilon_2}(z, y; t_2) = K_0(x, y; t_1 + t_2),$$

recovering the standard heat-kernel semigroup. A concrete failure mode is immediate: if $t + \varepsilon \le 0$, the Gaussian normalization is undefined/non-integrable, and composition no longer closes in the admissible family. This gives a model-level realization of the manuscript's compatibility logic: closure requires explicit admissibility conditions on the refinement-regulator pair.

Derivation 10.9 (First-order potential perturbation: composition closure to $O(V)$). Let $V : \mathbb{R}^d \to \mathbb{R}$ be bounded, and define the first-order regulated kernel by Duhamel expansion:

$$K^{(1)}_{\varepsilon,V}(x, y; t) := K_\varepsilon(x, y; t) - \int_0^t d\tau \int_{\mathbb{R}^d} d^d z\; K_\varepsilon(x, z; t-\tau)\, V(z)\, K_\varepsilon(z, y; \tau).$$

Using Derivation 10.8 for the free part and retaining terms through $O(V)$,

$$\int d^d z\; K^{(1)}_{\varepsilon,V}(x, z; t)\, K^{(1)}_{\varepsilon,V}(z, y; s) = K^{(1)}_{\varepsilon,V}(x, y; t + s) + O(V^2).$$

The first-order term combines by splitting $[0, t+s]$ into $[0,t]\cup[t,t+s]$: in the second interval set $\tau = t+\sigma$, use free-kernel composition to collapse intermediate integrations, and recover the same single-time-convolution structure at total time $t+s$. For distributional or too-singular attractive potentials, the convolution integral can diverge and the first-order kernel is not well-defined without extra renormalization/boundary data — precisely the regime where Appendix 10.5 becomes necessary.

Proposition 10.10 (Quantitative remainder and composition-defect bounds for bounded V). Let $H_0 = -\frac{\hbar^2}{2}\Delta$, $T_0(t) = e^{-tH_0}$, and $T_V(t) = e^{-t(H_0+V)}$ on $L^2(\mathbb{R}^d)$, with V a bounded multiplication operator (H_0+V is self-adjoint on $H^2(\mathbb{R}^d)$ by Kato–Rellich) and $M := \|V\|_\infty$. Define the first-order Duhamel approximation

$$T_V^{(1)}(t) := T_0(t) - \int_0^t T_0(t-\tau)\, V\, T_0(\tau)\, d\tau.$$

Then: 1. **Remainder bound**

$$\|T_V(t) - T_V^{(1)}(t)\| \le e^{Mt} - 1 - Mt \le \frac{1}{2}M^2t^2e^{Mt}.$$

2. **Composition-defect bound (first-order approximation)**

$$\|T_V^{(1)}(t)T_V^{(1)}(s) - T_V^{(1)}(t+s)\| \le C\, M^2\, (t+s)^2 e^{M(t+s)}$$

for a universal constant C (e.g. $C = 1$, which is sharp).

Proof sketch. Expand T_V by Duhamel to second order, bound iterated integrals with semigroup norms $\|T_0(r)\| \le 1$, $\|T_V(r)\| \le e^{Mr}$, and use

$$T_V^{(1)}(t)T_V^{(1)}(s) - T_V^{(1)}(t+s) = (T_V^{(1)}(t) - T_V(t))T_V^{(1)}(s) + T_V(t)(T_V^{(1)}(s) - T_V(s)) + (T_V(t+s) - T_V^{(1)}(t+s)).$$

Thus Derivation 10.9 has explicit quantitative control in the bounded-potential regime, not only formal $O(V)$ bookkeeping.

Derivation 10.10 (Exact nontrivial semigroup witness: Euclidean harmonic oscillator). For confining quadratic potential $V(x) = \frac{1}{2}\omega^2|x|^2$ with $\omega > 0$, the Euclidean kernel in $\mathbb{R}^d$ is the Mehler form

$$K_\omega(x,y;t) = \left(\frac{\omega}{2\pi\sinh(\omega t)}\right)^{d/2} \exp\left[-\frac{\omega}{2\sinh(\omega t)}\Big((|x|^2+|y|^2)\cosh(\omega t) - 2x\cdot y\Big)\right].$$

Gaussian integration in the intermediate variable gives, for $t_1, t_2 > 0$,

$$\int_{\mathbb{R}^d} d^d z\; K_\omega(x,z;t_1)K_\omega(z,y;t_2) = K_\omega(x,y;t_1+t_2),$$

using standard hyperbolic identities (notably $\sinh(a+b) = \sinh a\cosh b + \cosh a\sinh b$ and the induced identity for $\coth(a+b)$). Thus we obtain an all-order, nontrivial composition witness beyond the free and first-order perturbative regimes.

Defining $K_{\omega,\varepsilon}(x,y;t) := K_\omega(x,y;t+\varepsilon)$ with $\varepsilon > 0$, the same calculation yields additive regulator flow:

$$\int d^d z\; K_{\omega,\varepsilon_1}(x,z;t_1)K_{\omega,\varepsilon_2}(z,y;t_2) = K_{\omega,\varepsilon_1+\varepsilon_2}(x,y;t_1+t_2).$$

Since $K_{\omega,\varepsilon}(x,y;t) = K_\omega(x,y;t+\varepsilon)$ and $K_\omega(\cdot,y;t) \in L^1_x$ for all $t>0$, the same dominated convergence argument as in Proposition 10.9 gives $K_{\omega,\varepsilon} \to K_\omega$ pointwise and in L^1_x as $\varepsilon \to 0^+$, completing the regulated-removal chain for the nontrivial witness. This exact closure witness is specific to confining quadratic potentials ($\omega > 0$) in Euclidean time; inverted/nonconfining cases require separate treatment.

Two immediate sanity checks fix normalization and interpretation. In the free limit $\omega \to 0$, using $\sinh(\omega t) = \omega t + O((\omega t)^3)$ and $\cosh(\omega t) = 1 + O((\omega t)^2)$,

$$K_\omega(x,y;t) \xrightarrow[\omega\to 0]{} \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{|x-y|^2}{2t}\right) = K_0(x,y;t).$$

In the short-time limit $t \to 0^+$, $K_\omega(\cdot,\cdot;t)$ concentrates to δ in distributions, matching the semigroup initial condition. These checks ensure consistency with both the free witness (Derivation 10.8) and the standard heat-kernel normalization at short times.

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